Show your work and justify all answers.

For this assignment, we do not yet know anything about characteristic polynomials.

(12 pts)

(1) **[+2]** Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation with $T(\vec{e}_1) = \vec{0}$ and $T(\vec{e}_2) = \vec{e}_1$. Is T diagonalizable?

Solution: No. If T were diagonalizable, then there would be a basis $\{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{C}^2$ and scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $T(\vec{b}_i) = \lambda_i \vec{b}_i$. Notice that $M_T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so we are looking for solutions to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix},$$

for $a, b, \lambda \in \mathbb{C}$. We observe that this means that $\lambda a = b$ and $\lambda b = 0$. The latter tells us that either $\lambda = 0$ or $b = 0$.

If $\lambda = 0$, then $b = \lambda a = 0$, so, no matter what, $b = 0$. Thus, $\begin{bmatrix} a \\ b \end{bmatrix} \in \operatorname{span}\{\vec{e}_1\}$.

As such, there cannot be a basis $\{\vec{b}_1, \vec{b}_2\}$ for which $T(\vec{b}_i) = \lambda_i \vec{b}_i$ for some $\lambda_i \in \mathbb{C}$, so T cannot be diagonalizable. □

(2) **[+2]** Let $T, P\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations where T projects $\mathbb{R}^2$ onto the x-axis and $P\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $P\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find the matrix representation of $T \circ P$ with respect to the standard basis.

Solution: Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $P(\vec{b}_1) = 2\vec{b}_1$ and $P(\vec{b}_2) = \vec{b}_1 - \vec{b}_2$. Thus, $[M_P]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, so

$$M_P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}.$$

Now, certainly $M_T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so

$$M_{T \circ P} = M_T M_P = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}.$$

□

(3) Let $A \in \mathbb{C}^{n \times n}$ and suppose that $\vec{v} \in \mathbb{C}^n$ is an eigenvector for A with eigenvalue λ.

(a) **[+1]** Show that if $A^k = O_n$ for some positive integer k, then $\lambda = 0$.

Solution: We begin by noticing that $A^k \vec{v} = A^{k-1}(A\vec{v}) = \lambda A^{k-1} \vec{v} = \cdots = \lambda^k \vec{v}$. Thus, $\lambda^k \vec{v} = A^k \vec{v} = O_n \vec{v} = \vec{0}$. Since $\vec{v}$ is an eigenvector for A, we know that $\vec{v} \neq 0$, so this means that $\lambda^k = 0$, which happens if and only if $\lambda = 0$. □

(b) **[+1]** Show that if p is any polynomial, then $\vec{v}$ is an eigenvector for $p(A)$ with eigenvalue $p(\lambda)$.

Solution: Since $A^k\vec{v} = \lambda^k\vec{v}$, if $p(t) = \sum_{j=0}^{k} a_j t^j$ for some $a_j \in \mathbb{C}$, then

$$p(A)\vec{v} = \sum_{j=0}^{k} a_j A^j \vec{v} = \sum_{j=0}^{k} a_j \lambda^j \vec{v} = p(\lambda)\vec{v},$$

so $\vec{v}$ is an eigenvector for $p(A)$ with eigenvalue $p(\lambda)$. □

(c) **[+1]** Show that if A is unitary, then $|\lambda| = 1$. (Hint: consider Hermitian inner products)

Solution: We first compute $\langle A\vec{v}, A\vec{v}\rangle = \langle \lambda\vec{v}, \lambda\vec{v}\rangle = \overline{\lambda}\lambda\langle \vec{v}, \vec{v}\rangle = |\lambda|^2\|v\|^2$. On the other hand, since A is unitary, we have $\langle A\vec{v}, A\vec{v}\rangle = \langle \vec{v}, \vec{v}\rangle = \|v\|^2$. Since $\vec{v} \neq 0$, this implies that $|\lambda|^2 = 1$, so $|\lambda| = 1$. □

(4) **[+3]** Let V be any vector space over $\mathbb{C}$ and $T\colon V \to V$ be a linear transformation. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are *distinct* and that $v_1, \ldots, v_n \in V$ are non-zero vectors satisfying $T(v_i) = \lambda_i v_i$.

Prove that $\{v_1, \ldots, v_n\}$ is linearly independent. (Hint: induction on n)

Solution: Base case: $n = 1$. Since $v_1 \neq 0$, certainly $\{v_1\}$ is linearly independent.

Induction hypothesis: For some $N > 1$, if $v_1, \ldots, v_{N-1} \in V$ are non-zero vectors with $T(v_i) = \lambda_i v_i$ where $\lambda_1, \ldots, \lambda_{N-1} \in \mathbb{C}$ are distinct, then $\{v_1, \ldots, v_{N-1}\}$ is linearly independent.

Induction step: Suppose that $v_1, \ldots, v_N \in V$ are non-zero vectors with $T(v_i) = \lambda_i v_i$ where $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ are distinct.

Consider a linear combination $c_1v_1 + \cdots + c_Nv_N = 0$ where $c_i \in \mathbb{C}$. Applying the transformation T to both sides, we also find that $0 = T(c_1v_1 + \cdots + c_Nv_N) = c_1\lambda_1v_1 + \cdots + c_N\lambda_Nv_N$. Now, take this equation and subtract from it λ_N times the original $c_1v_1 + \cdots + c_Nv_N = 0$ to find

$$0 = c_1(\lambda_1 - \lambda_N)v_1 + c_2(\lambda_1 - \lambda_N)v_2 + \cdots + c_N(\lambda_N - \lambda_N)v_N = c_1(\lambda_1 - \lambda_N)v_1 + \cdots + c_{N-1}(\lambda_{N-1} - \lambda_N)v_{N-1}$$

By the induction hypothesis, $\{v_1, \ldots, v_{N-1}\}$ is linearly independent, so this implies that $c_i(\lambda_i - \lambda_{N-1}) = 0$ for all $i \in [N-1]$. Now, by assumption, the λ_i's are distinct, so $\lambda_i - \lambda_N \neq 0$ for all $i \in [N-1]$, so we must have $c_i = 0$ for all $i \in [N-1]$. Hence,

$$0 = c_1v_1 + \cdots + c_Nv_N = c_Nv_N,$$

so since $v_N \neq 0$ by assumption, we indeed have $c_1 = \cdots = c_N = 0$, so $\{v_1, \ldots, v_N\}$ is linearly independent. □

(5) **[+2]** Show that $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if A is unitarily similar to a diagonal matrix with real entries; that is, $A = UDU^*$ where U is unitary and $D \in \mathbb{R}^{n\times n}$ is diagonal.

Solution: First suppose that $A = UDU^*$ where U is unitary and $D \in \mathbb{R}^{n\times n}$ is diagonal. Since $D \in \mathbb{R}^{n\times n}$ is diagonal, we have $D^* = D$, so $A^* = (UDU^*)^* = UD^*U^* = UDU^* = A$.

On the other hand, suppose that A is Hermitian. By Schur's triangularization theorem, we can write $A = UTU^*$ where U is unitary and T is upper-triangular. Since U is unitary, we can isolate $T = U^*AU$. Thus, since A is Hermitian, $T^* = (U^*AU)^* = U^*A^*U = U^*AU$, so T is Hermitian as well. Thus, $T_{ii} = \overline{T_{ii}}$, so $T_{ii} \in \mathbb{R}$. Also, since T is upper-triangular, we know that $T_{ij} = 0$ for all $i > j$, but since T is Hermitian, we then have for all $i < j$, $T_{ij} = \overline{T_{ji}} = 0$ as well. Thus, T is a real, diagonal matrix. □

(6) **Bonus[+1]**[1] Let $T_1, \ldots, T_n \in \mathbb{C}^{n\times n}$ be upper-triangular matrices. Show that if $(T_i)_{ii} = 0$ for all $i \in [n]$, then $T_1 T_2 \cdots T_n = O_n$.

Solution: We prove the following statement by induction on k: For $k \in [n]$, the first k columns of $T_1 \cdots T_k$ are all 0's, from which the problem statement will follow.

Base case: $k = 1$. T_1 is upper triangular and $(T_1)_{11} = 0$, so the first column of T_1 is all 0's.

Induction hypothesis: For some $1 < K \leq n$, we have that the first $K - 1$ columns of $T_1 \cdots T_{K-1}$ are all 0's.

Induction step: Set $R = T_1 \cdots T_{K-1}$, so $T_1 \cdots T_K = RT_K$. By the induction hypothesis, we know that the first $K - 1$ columns of R are all 0's. Additionally, we know that T_K is upper triangular and $(T_K)_{KK} = 0$.

Fix $i \in [n]$ and $j \in [K]$; we need to show that $(RT_K)_{ij} = 0$. We compute

$$(RT_K)_{ij} = \sum_{\ell=1}^{n} R_{i\ell}(T_K)_{\ell j} = \sum_{\ell=1}^{K-1} R_{i\ell}(T_K)_{\ell j} + \sum_{\ell=K}^{n} R_{i\ell}(T_K)_{\ell j}.$$

Now, for all $\ell \in [K - 1]$, we know that $R_{i\ell} = 0$, so $(RT_K)_{ij} = \sum_{\ell=K}^{n} R_{i\ell}(T_K)_{\ell j}$. Since T_K is upper-triangular, we know that $(T_K)_{\ell j} = 0$ for all $\ell > j$, and since additionally $(T_K)_{KK} = 0$ by assumption, we in fact know that $(T_K)_{\ell j} = 0$ for all $\ell \geq K$ since $j \in [K]$.

Thus, $(RT_K)_{ij} = 0$ for all $i \in [n]$ and $j \in [K]$, so the first K columns of $T_1 \cdots T_K$ are all 0's as needed.

□

[1] We will need to use this result in lecture, so even if you do not solve this problem, at least read the proof once it is posted.