Show your work and justify all answers.

For this assignment, we do not yet know anything about characteristic polynomials.

(12 pts)

(1) [**+2**] Let $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation with $T(\vec{e}_1) = \vec{0}$ and $T(\vec{e}_2) = \vec{e}_1$. Is T diagonalizable?

(2) [**+2**] Let $T, P\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations where T projects $\mathbb{R}^2$ onto the x-axis and $P\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}$ and $P\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$. Find the matrix representation of $T \circ P$ with respect to the standard basis.

(3) Let $A \in \mathbb{C}^{n\times n}$ and suppose that $\vec{v} \in \mathbb{C}^n$ is an eigenvector for A with eigenvalue λ.

   (a) [**+1**] Show that if $A^k = O_n$ for some positive integer k, then $\lambda = 0$.

   (b) [**+1**] Show that if p is any polynomial, then $\vec{v}$ is an eigenvector for $p(A)$ with eigenvalue $p(\lambda)$.

   (c) [**+1**] Show that if A is unitary, then $|\lambda| = 1$. (Hint: consider Hermitian inner products)

(4) [**+3**] Let V be any vector space over $\mathbb{C}$ and $T\colon V \to V$ be a linear transformation. Suppose that $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are *distinct* and that $v_1, \ldots, v_n \in V$ are non-zero vectors satisfying $T(v_i) = \lambda_i v_i$.

   Prove that $\{v_1, \ldots, v_n\}$ is linearly independent. (Hint: induction on n)

(5) [**+2**] Show that $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if A is unitarily similar to a diagonal matrix with real entries; that is, $A = UDU^*$ where U is unitary and $D \in \mathbb{R}^{n\times n}$ is diagonal.

(6) **Bonus**[**+1**][1] Let $T_1, \ldots, T_n \in \mathbb{C}^{n\times n}$ be upper-triangular matrices. Show that if $(T_i)_{ii} = 0$ for all $i \in [n]$, then $T_1 T_2 \cdots T_n = O_n$.

[1]We will need to use this result in lecture, so even if you do not solve this problem, at least read the proof once it is posted.