Show your work and justify all answers.

(12 pts)

(1) Let U, V, W be vector spaces (over $\mathbb{C}$) and let $L\colon U \to V$ and $R\colon V \to W$ be linear transformations.

(a) **[+1]** Prove that $\ker(R \circ L) \supseteq \ker L$ and that $\operatorname{im}(R \circ L) \subseteq \operatorname{im} R$.

Solution: If $u \in \ker L$, then $L(u) = 0$, so $(R \circ L)(u) = R(0) = 0$, so $u \in \ker(R \circ L)$.
If $w \in \operatorname{im}(R \circ L)$, then there is some $u \in U$ with $w = (R \circ L)(u) = R(L(u))$. Since $L(u) \in V$, there is some $v \in V$ (namely $v = L(u)$) with $R(v) = u$; thus $u \in \operatorname{im} R$. □

(b) **[+2]** Prove that $\ker(R \circ L) = \ker L$ if and only if $\operatorname{im} L \cap \ker R = \{0\}$.

Solution: ($\Rightarrow$) Suppose that $v \in \operatorname{im} L \cap \ker R$; we need to show that $v = 0$. Since $v \in \operatorname{im} L$, we can find some $u \in U$ with $L(u) = v$. Furthermore, since $v \in \ker R$, we know that $0 = R(v) = R(L(u)) = (R \circ L)(u)$, so $u \in \ker(R \circ L)$. But since $\ker(R \circ L) = \ker L$ by assumption, we see that $u \in \ker L$, so $v = L(u) = 0$.
($\Leftarrow$) By part (a), we know that $\ker L \subseteq \ker(R \circ L)$, so we need only show the reverse inclusion. Let $u \in \ker(R \circ L)$, so $(R \circ L)(u) = 0$; we need to show that $u \in \ker L$. Since $0 = (R \circ L)(u) = R(L(u))$, we see that $L(u) \in \ker R$. On the other hand, certainly $L(u) \in \operatorname{im} L$, so $L(u) \in \operatorname{im} L \cap \ker R = \{0\}$, so $L(u) = 0$. Thus, $u \in \ker L$ as needed. □

(c) **[+2]** Prove that $\operatorname{im}(R \circ L) = \operatorname{im} R$ if and only if $\ker R + \operatorname{im} L = V$.
(Hint: it may be helpful to recall that $R(a) = R(b)$ if and only if $a - b \in \ker R$)

Solution: ($\Rightarrow$) Fix $v \in V$; we need to find $a \in \ker R$ and $b \in \operatorname{im} L$ such that $a+b = v$. Set $w = R(v)$, so $w \in \operatorname{im} R$. Since $\operatorname{im} R = \operatorname{im}(R \circ L)$, we can find $u \in U$ with $(R \circ L)(u) = w$. Set $b = L(u)$, so $b \in \operatorname{im} L$. Finally, set $a = v - b$, so $R(a) = R(v - b) = R(v) - R(b) = R(v) - R(L(u)) = w - w = 0$. Thus $a \in \ker R$, $b \in \operatorname{im} L$ and $a + b = v$.
($\Leftarrow$) By part (a), we know that $\operatorname{im}(R \circ L) \subseteq \operatorname{im} R$, so we need only show the reverse inclusion. Let $w \in \operatorname{im} R$; we need to find $u \in U$ for which $(R \circ L)(u) = w$. Since $w \in \operatorname{im} R$, we can find $v \in V$ with $R(v) = w$. Now, by assumption, $\ker R + \operatorname{im} L = V$, so we can find $a \in \ker R$ and $b \in \operatorname{im} L$ with $a + b = v$. Since $b \in \operatorname{im} L$, we can then find $u \in U$ with $L(u) = b$. Now, since $a \in \ker R$, we find that $(R \circ L)(u) = R(b) = 0 + R(b) = R(a) + R(b) = R(a + b) = R(v) = w$, so $w \in \operatorname{im}(R \circ L)$. □

(2) **[+3]** Let U, V be finite-dimensional vector spaces (over $\mathbb{C}$) and let $L\colon U \to V$ be a linear transformation. Consider the following three statements:

1. L is injective.
2. L is surjective.
3. $\dim U = \dim V$.

Prove that any two of the above statements imply the third; e.g. show that if L is injective and $\dim U = \dim V$, then L is surjective, etc.
(1 point for each implication.)

Solution: ($1 + 2 \implies 3$) Since L is both injective and surjective, we know that L is bijective, so U and V are isomorphic. We've seen in class that this implies that $\dim U = \dim V$.

($1+3 \implies 2$) Consider the rank–nullity theorem: $\dim\ker L + \dim\operatorname{im} L = \dim U$. Since L is injective, we know that $\dim\ker L = 0$, so $\dim\operatorname{im} L = \dim U = \dim V$. Since $\operatorname{im} L \leq V$ and V is finite-dimensional, this means that $\operatorname{im} L = V$, so L is surjective.

($2+3 \implies 1$) Again consider the rank–nullity theorem. Since L is surjective, we know that $\operatorname{im} L = V \implies \dim\operatorname{im} L = \dim V = \dim U$, so $\dim\ker L + \dim\operatorname{im} L = \dim U \implies \dim\ker L = 0$. Thus, $\ker L = \{0\}$, and we've seen in class that this implies that L is injective. □

(3) Let V be a finite-dimensional inner product space (over $\mathbb{C}$) and let $S \leq V$.

(a) **[+2]** Prove that if $L\colon V \to V$ is a linear transformation with $L(s) = s$ for all $s \in S$ and $L(t) = 0$ for all $t \in S^\perp$, then $L = \operatorname{proj}_S$.[1]
(Hint: linear extension lemma)

Solution: Since L and proj_S are both linear transformations, in order to show that $L = \operatorname{proj}_S$, through the linear extension lemma, it suffices to show that there is a basis $\mathcal{B}$ for V such that $L(b) = \operatorname{proj}_S b$ for all $b \in \mathcal{B}$.
Suppose that $\dim V = n$ and $\dim S = k$, so by Gram–Schmidt, we can find $\{v_1, \ldots, v_k\}$, an orthonormal basis for S, and $\{v_{k+1}, \ldots, v_n\}$, an orthonormal basis for $S^\perp$; hence $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V.
Now, for all $i \in [k]$, we know that $v_i \in S$ and $v_i - v_i = 0 \in S^\perp$, so $\operatorname{proj}_S v_i = v_i = L(v_i)$. On the other hand, for all $i \in \{k+1, \ldots, n\}$, we know that $0 \in S$ and $v_i - 0 = v_i \in S^\perp$, so $\operatorname{proj}_S v_i = 0 = L(v_i)$. Thus, proj_S and L agree on a basis for V, so $\operatorname{proj}_S = L$. □

(b) **[+2]** Consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product and let $\{\vec{s}_1, \ldots, \vec{s}_k\}$ be *any* basis for $S \leq \mathbb{C}^n$. In DSW3, we showed that if $A = \begin{bmatrix} \vec{s}_1 & \cdots & \vec{s}_k \end{bmatrix}$, then $\operatorname{proj}_S = A(A^*A)^{-1}A^*$. Give an alternative proof of this fact by using part (a).

Solution: Since $A(A^*A)^{-1}A^* \in \mathbb{C}^{n\times n}$, we indeed know that $A(A^*A)^{-1}A^*\colon \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation. We need to show that if $\vec{s} \in S$, then $A(A^*A)^{-1}A^*\vec{s} = \vec{s}$ and if $\vec{t} \in S^\perp$, then $A(A^*A)^{-1}A^*\vec{t} = \vec{0}$.
First, let $\vec{s} \in S$, so there is some $\vec{x} \in \mathbb{C}^k$ for which $A\vec{x} = \vec{s}$. Thus,

$$A(A^*A)^{-1}A^*\vec{s} = A(A^*A)^{-1}A^*A\vec{x} = AI_k\vec{x} = A\vec{x} = \vec{s},$$

as needed. Now, let $\vec{t} \in S^\perp$, so $\langle \vec{s}, \vec{t}\rangle = 0$ for all $\vec{s} \in S$. Notice that $A^*\vec{t} = \begin{bmatrix} \langle \vec{s}_1, \vec{t}\rangle \\ \vdots \\ \langle \vec{s}_k, \vec{t}\rangle \end{bmatrix}$, so $A(A^*A)^{-1}A^*\vec{t} = A(A^*A)^{-1}\vec{0} = \vec{0}$, as needed.
Thus, $\operatorname{proj}_S = A(A^*A)^{-1}A^*$ by part (a). □

[1]Recall that $\operatorname{proj}_S v = p$ if $p \in S$ and $v - p \in S^\perp$.