Show your work and justify all answers.

(12 pts)

(1) Let U, V, W be vector spaces (over $\mathbb{C}$) and let $L\colon U \to V$ and $R\colon V \to W$ be linear transformations.

(a) [**+1**] Prove that $\ker(R \circ L) \supseteq \ker L$ and that $\operatorname{im}(R \circ L) \subseteq \operatorname{im} R$.

(b) [**+2**] Prove that $\ker(R \circ L) = \ker L$ if and only if $\operatorname{im} L \cap \ker R = \{0\}$.

(c) [**+2**] Prove that $\operatorname{im}(R \circ L) = \operatorname{im} R$ if and only if $\ker R + \operatorname{im} L = V$.

(Hint: it may be helpful to recall that $R(a) = R(b)$ if and only if $a - b \in \ker R$)

(2) [**+3**] Let U, V be finite-dimensional vector spaces (over $\mathbb{C}$) and let $L\colon U \to V$ be a linear transformation. Consider the following three statements:

1. L is injective.
2. L is surjective.
3. $\dim U = \dim V$.

Prove that any two of the above statements imply the third; e.g. show that if L is injective and $\dim U = \dim V$, then L is surjective, etc.

(1 point for each implication.)

(3) Let V be a finite-dimensional inner product space (over $\mathbb{C}$) and let $S \leq V$.

(a) [**+2**] Prove that if $L\colon V \to V$ is a linear transformation with $L(s) = s$ for all $s \in S$ and $L(t) = 0$ for all $t \in S^\perp$, then $L = \operatorname{proj}_S$.[1]

(Hint: linear extension lemma)

(b) [**+2**] Consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product and let $\{\vec{s}_1, \ldots, \vec{s}_k\}$ be *any* basis for $S \leq \mathbb{C}^n$. In DSW3, we showed that if $A = \begin{bmatrix} \vec{s}_1 & \cdots & \vec{s}_k \end{bmatrix}$, then $\operatorname{proj}_S = A(A^*A)^{-1}A^*$. Give an alternative proof of this fact by using part (a).

[1] Recall that $\operatorname{proj}_S v = p$ if $p \in S$ and $v - p \in S^\perp$.