Show your work and justify all answers.

(10 pts)

(1) **[+2]** Let V be an inner product space over $\mathbb{C}$. For subspaces $S_1, S_2 \leq V$, we write $S_1 \perp S_2$ if $\langle s_1, s_2 \rangle = 0$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Suppose that $S_1, \ldots, S_n \leq V$ are finite-dimensional subspaces such that $S_i \perp S_j$ for all $i \neq j \in [n]$. Prove that $\dim\big(S_1 + \cdots + S_n\big) = \dim S_1 + \cdots + \dim S_n$.

Solution: By Gram–Schmidt, we can find an orthonormal basis $\mathcal{B}_i$ for S_i. We claim that $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ (where repetition is allowed[1]) is a basis for $S_1 + \cdots + S_n$ which will prove the claim.

By the proof of problem (4) in DSW2, we know that $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ spans $S_1 + \cdots + S_n$, so we need only argue linear independence. In fact, we will argue that $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ is orthonormal, which is even better!

By assumption, we already know that if $b \in \mathcal{B}_1 \cup \cdots \cup B_n$, then $\|b\| = 1$. Now, let $a \neq b \in \mathcal{B}_1 \cup \cdots \cup B_n$. If $a, b \in \mathcal{B}_i$ for some i, then we already know that $\langle a, b \rangle = 0$ since $\mathcal{B}_i$ is orthonormal by assumption. On the other hand, if $a \in \mathcal{B}_i$ and $b \in \mathcal{B}_j$ for some $i \neq j$, then $\langle a, b \rangle = 0$ since $S_i \perp S_j$ for all $i \neq j$ by assumption. □

(2) Let V be an inner product space over $\mathbb{C}$ and let $S, T \leq V$.

(a) **[+2]** Prove that $(S+T)^\perp = S^\perp \cap T^\perp$.

Solution: ($\supseteq$) Let $x \in S^\perp \cap T^\perp$ so we know that for any $s \in S, t \in T$, we have $\langle x, s \rangle = \langle x, t \rangle = 0$. Thus, for any $s + t \in S + T$, we know that $\langle x, s + t \rangle = \langle x, s \rangle + \langle x, t \rangle = 0$, so $x \in (S+T)^\perp$.
($\subseteq$) Let $x \in (S+T)^\perp$, so for any $y \in S+T$, we know that $\langle x, y \rangle = 0$. Now, fix any $s \in S$. Since $0 \in T$, we know that $s \in S + T$, so $\langle x, s \rangle = 0$, i.e. $x \in S^\perp$. Similarly, $x \in T^\perp$, so $x \in S^\perp \cap T^\perp$. □

(b) **[+2]** Prove that if V is finite-dimensional, then $(S \cap T)^\perp = S^\perp + T^\perp$.

Solution: Set $S_1 = S^\perp$ and $T_1 = T^\perp$. Since V is finite-dimensional we know that $S_1^\perp = S$ and $T_1^\perp = T$.
By part (a), we know that $(S_1 + T_1)^\perp = S_1^\perp \cap T_1^\perp$ so by taking the orthogonal complement of both sides and using the fact that $(S_1 + T_1)^{\perp\perp} = S_1 + T_1$, we find that

$$S^\perp + T^\perp = S_1 + T_1 = (S_1^\perp \cap T_1^\perp)^\perp = (S \cap T)^\perp.$$

□

(3) This exercise will walk through a classification of all inner products on $\mathbb{C}^n$.

A matrix $A \in \mathbb{C}^{n \times n}$ is called *positive-definite* if A is Hermitian[2] and $\vec{x}^* A \vec{x} \geq 0$ for all $\vec{x} \in \mathbb{C}^n$ with $\vec{x}^* A \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$. We write $A \succ 0$ to mean that A is positive-definite.

Finally, for a matrix $A \in \mathbb{C}^{n \times n}$, define $\langle \vec{x}, \vec{y} \rangle_A = \vec{x}^* A \vec{y}$.

(a) **[+2]** Show that if $A \succ 0$, then $\langle \vec{x}, \vec{y} \rangle_A$ is an inner product on $\mathbb{C}^n$. (Note that if $c \in \mathbb{C}$, then $\overline{c} = c^*$.)

Solution: We verify the three criteria to be an inner product.

[1] Though, the proof will show this does not happen

[2] Recall that this means $A^* = A$.

1. We have $\overline{\langle \vec{x}, \vec{y} \rangle_A} = \overline{\vec{x}^* A \vec{y}} = (\vec{x}^* A \vec{y})^* = \vec{y}^* A^* \vec{x} = \vec{y}^* A \vec{x} = \langle \vec{y}, \vec{x} \rangle_A$ since $A^* = A$ and $\vec{x}^* A \vec{y} \in \mathbb{C}$.
2. For $\vec{y}, \vec{z} \in \mathbb{C}^n$ and $c, d \in \mathbb{C}$, we have $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle_A = \vec{x}^* A (c\vec{y} + d\vec{z}) = c\vec{x}^* A \vec{y} + d\vec{x}^* A \vec{z} = c\langle \vec{x}, \vec{y} \rangle_A + d\langle \vec{x}, \vec{z} \rangle_A$.
3. This is the whole point of the definition of positive-definite! $\langle \vec{x}, \vec{x} \rangle_A = \vec{x}^* A \vec{x} \geq 0$ with equality if and only if $\vec{x} = \vec{0}$ since $A \succ 0$.

$\square$

(b) **[+2]** Let $\langle \cdot, \cdot \rangle$ be *any* inner product on $\mathbb{C}^n$. Define the matrix $A \in \mathbb{C}^{n \times n}$ by $A_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$. Prove that $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle_A$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$ and that $A \succ 0$.

Solution: Note right away that $\overline{A_{ij}} = \overline{\langle \vec{e}_i, \vec{e}_j \rangle} = \langle \vec{e}_j, \vec{e}_i \rangle = A_{ji}$, so $A^* = A$.
We now claim that for any $\vec{x}, \vec{y} \in \mathbb{C}^n$, we have $\langle \vec{x}, \vec{y} \rangle = \vec{x}^* A \vec{y}$. Indeed, write $\vec{x} = x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n$ and $\vec{y} = y_1 \vec{e}_1 + \cdots + y_n \vec{e}_n$. Then, by linearity and conjugate-linearity of $\langle \cdot, \cdot \rangle$, we have

$$\langle \vec{x}, \vec{y} \rangle = \left\langle \sum_{i=1}^{n} x_i \vec{e}_i, \ \sum_{j=1}^{n} y_j \vec{e}_j \right\rangle = \sum_{i,j} \overline{x_i} y_j \langle \vec{e}_i, \vec{e}_j \rangle = \sum_{i,j} A_{ij} \overline{x_i} y_j = \vec{x}^* A \vec{y}.$$

Therefore, $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle_A$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.
The fact that $A \succ 0$ follows immediately from the fact that $\vec{x}^* A \vec{x} = \langle x, x \rangle \geq 0$ with equality if and only if $\vec{x} = \vec{0}$ since $\langle \cdot, \cdot \rangle$ was assumed to be an inner product. $\square$