Show your work and justify all answers.

(10 pts)

(1) **[+2]** Let V be an inner product space over $\mathbb{C}$. For subspaces $S_1, S_2 \leq V$, we write $S_1 \perp S_2$ if $\langle s_1, s_2 \rangle = 0$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Suppose that $S_1, \ldots, S_n \leq V$ are finite-dimensional subspaces such that $S_i \perp S_j$ for all $i \neq j \in [n]$. Prove that $\dim\big(S_1 + \cdots + S_n\big) = \dim S_1 + \cdots + \dim S_n$.

(2) Let V be an inner product space over $\mathbb{C}$ and let $S, T \leq V$.

(a) **[+2]** Prove that $(S + T)^\perp = S^\perp \cap T^\perp$.

(b) **[+2]** Prove that if V is finite-dimensional, then $(S \cap T)^\perp = S^\perp + T^\perp$.

(3) This exercise will walk through a classification of all inner products on $\mathbb{C}^n$.

A matrix $A \in \mathbb{C}^{n\times n}$ is called *positive-definite* if A is Hermitian[1] and $\vec{x}^* A \vec{x} \geq 0$ for all $\vec{x} \in \mathbb{C}^n$ with $\vec{x}^* A \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$. We write $A \succ 0$ to mean that A is positive-definite.

Finally, for a matrix $A \in \mathbb{C}^{n\times n}$, define $\langle \vec{x}, \vec{y} \rangle_A = \vec{x}^* A \vec{y}$.

(a) **[+2]** Show that if $A \succ 0$, then $\langle \vec{x}, \vec{y} \rangle_A$ is an inner product on $\mathbb{C}^n$. (Note that if $c \in \mathbb{C}$, then $\bar{c} = c^*$.)

(b) **[+2]** Let $\langle \cdot, \cdot \rangle$ be *any* inner product on $\mathbb{C}^n$. Define the matrix $A \in \mathbb{C}^{n\times n}$ by $A_{ij} = \langle \vec{e}_i, \vec{e}_j \rangle$. Prove that $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle_A$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$ and that $A \succ 0$.

[1] Recall that this means $A^* = A$.