Show your work and justify all answers.

## (11 pts)

(1) This exercise will walk through a proof that not every norm is associated with an inner product.

(a) **[+1]** Let $V$ be an inner product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and let $\| \cdot \|$ be the associated norm. Prove that for any $x, y \in V$, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big).$$

(This is known as the parallelogram rule)

**Solution:** We compute

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x - y, x - y\rangle \\ &= \big(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\big) + \big(\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle\big) \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle = 2\big(\|x\|^2 + \|y\|^2\big).\end{aligned}$$

□

(b) **[+1]** For a vector $\vec{x} \in \mathbb{C}^n$, the $\ell_1$-norm of $\vec{x}$ is defined as $\|\vec{x}\|_1 = \sum_{k=1}^n |x_k|$. Prove that $\| \cdot \|_1$ is indeed a norm on $\mathbb{C}^n$.

**Solution:** We verify the three properties.

1. For $\vec{x} \in \mathbb{C}^n$, we observe that $|x_k| \in \mathbb{R}_+$, so $\|\vec{x}\|_1 \in \mathbb{R}_+$ as well. Furthermore, if $\|\vec{x}\|_1 = 0$, then we must have $|x_k| = 0$ for all $k \in [n]$, so $\vec{x} = \vec{0}$.
2. Let $\vec{x} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. We have $\|\alpha\vec{x}\|_1 = \sum_{k=1}^n |\alpha x_k| = |\alpha| \sum_{k=1}^n |x_k| = |\alpha|\|\vec{x}\|_1$.
3. We know that for $c, d \in \mathbb{C}$ we have $|c + d| \le |c| + |d|$, so for $\vec{x}, \vec{y} \in \mathbb{C}^n$, we have

$$\|x + y\|_1 = \sum_{k=1}^n |x_k + y_k| \le \sum_{k=1}^n |x_k| + \sum_{k=1}^n |y_k| = \|x\|_1 + \|y\|_1.$$

□

(c) **[+2]** Show that for $n \ge 2$, there is no inner product on $\mathbb{C}^n$ associated with the $\ell_1$-norm.

**Solution:** We show that $\| \cdot \|_1$ does not satisfy the parallelogram rule. We notice that $\|\vec{e}_1\|_1 = \|\vec{e}_2\|_1 = 1$ and $\|\vec{e}_1 + \vec{e}_2\|_1 = \|\vec{e}_1 - \vec{e}_2\|_1 = 2$.
Thus, $\|\vec{e}_1 + \vec{e}_2\|_1^2 + \|\vec{e}_1 - \vec{e}_2\|_1^2 = 8$ while $2\big(\|\vec{e}_1\|_1^2 + \|\vec{e}_2\|_1^2\big) = 4$. □

(2) **[+2]** Let $\mathcal{P}_n$ denote the space of polynomials of degree at most $n$ with real coefficients as a vector space over $\mathbb{R}$. Equip $\mathcal{P}_n$ with the inner product $\langle p, q\rangle = \int_{-1}^1 p(x)q(x)dx$. (While you do not have to prove it, convince yourself that $\langle \cdot, \cdot \rangle$ is indeed an inner product on $\mathcal{P}_n$)

Use the Gram–Schmidt algorithm to find an orthonormal basis for $\mathcal{P}_2$ by starting with the basis $\{1, x, x^2\}$.

**Solution:** Note that $\|1\|^2 = \int_{-1}^1 1^2 dx = 2$, so we can start with $v_1 = \frac{1}{\|1\|} \cdot 1 = \frac{1}{\sqrt{2}}$.

Now, set

$$v_2' = x - \langle x, v_1\rangle v_1 = x - \frac{1}{\sqrt{2}} \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot x dx = x.$$

Since $\|v_2'\|^2 = \int_{-1}^1 x^2 dx = \frac{2}{3}$, we can thus set

$$v_2 = \frac{1}{\|v_2'\|} v_2' = \sqrt{\frac{3}{2}} x.$$

Finally, set

$$v_3' = x^2 - \langle x^2, v_1 \rangle v_1 - \langle x^2, v_2 \rangle v_2 = x^2 - \frac{1}{\sqrt{2}} \int_{-1}^1 \frac{1}{\sqrt{2}} \cdot x^2 dx - \sqrt{\frac{3}{2}} x \int_{-1}^1 \sqrt{\frac{3}{2}} x \cdot x^2 dx = x^2 - \frac{1}{3}.$$

Since

$$\|v_3'\|^2 = \int_{-1}^1 \left( x^2 - \frac{1}{3} \right)^2 dx = \frac{8}{45},$$

we finally find

$$v_3 = \frac{1}{\|v_3'\|} v_3' = \sqrt{\frac{45}{8}} \left( x^2 - \frac{1}{3} \right).$$

Therefore, an orthogonal basis for $\mathcal{P}_2$ is

$$\left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} x, \frac{3\sqrt{5}}{2\sqrt{2}} x^2 - \frac{\sqrt{5}}{2\sqrt{2}} \right\}.$$

□

(3) Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^n$.

(a) (Not graded. This is just a good fact to keep in mind.[1]) Fix $A, B \in \mathbb{C}^{m \times n}$. Show that $A = B$ if and only if $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{C}^n$.

**Solution:** Certainly if $A = B$, then $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{C}^n$, so we focus only on the opposite direction. For this direction, notice that for $i \in [n]$, we have that $A\vec{e}_i$ is the $i$th column of $A$. Thus, since $A\vec{e}_i = B\vec{e}_i$ for every $i \in [n]$, we know that the $i$th column of $A$ is the same as the $i$th column of $B$, i.e. $A = B$. □

(b) **[+2]** Fix $A, B \in \mathbb{C}^{n \times n}$. Show that $A = B$ if and only if $\langle \vec{x}, A\vec{y} \rangle = \langle \vec{x}, B\vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

**Solution:** Obviously if $A = B$, then $\langle \vec{x}, A\vec{y} \rangle = \langle \vec{x}, B\vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, so we focus only on the other direction.

There are a number of ways to do this[2], but we will take the hint from part (a) in that we will show that $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{C}^n$. Fix $\vec{x} \in \mathbb{C}^n$ and compute

$$\langle A\vec{x} - B\vec{x}, A\vec{x} - B\vec{x} \rangle = \langle A\vec{x} - B\vec{x}, A\vec{x} \rangle - \langle A\vec{x} - B\vec{x}, B\vec{x} \rangle = 0,$$

since $\langle \vec{y}, A\vec{x} \rangle = \langle \vec{y}, B\vec{x} \rangle$ for any $\vec{x}, \vec{y} \in \mathbb{C}^n$. Therefore, $A\vec{x} - B\vec{x} = \vec{0}$, so $A\vec{x} = B\vec{x}$. Since $\vec{x} \in \mathbb{C}^n$ was arbitrary, this implies that $A = B$ by part (a). □

(c) **[+1]** Show that if $A \in \mathbb{C}^{n \times n}$, then for any $\vec{x}, \vec{y} \in \mathbb{C}^n$, we have $\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle$.

**Solution:** We compute $\langle A\vec{x}, \vec{y} \rangle = (A\vec{x})^* \vec{y} = \vec{x}^* A^* \vec{y} = \langle \vec{x}, A^*\vec{y} \rangle$. □

(d) **[+1]** A matrix $A \in \mathbb{C}^{n \times n}$ is called *Hermitian* if $A^* = A$. Show that $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if $\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A\vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

**Solution:** By part (c), we know that $\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle$ for any $\vec{x}, \vec{y} \in \mathbb{C}^n$. Thus, by part (b), we see that $\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A\vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$ if and only if $A = A^*$. □

[1] wink wink nudge nudge

[2] Some of these other ways are arguably simpler than this one, but I, personally, find this proof to be the most enlightening, especially since it works for *any* inner product on $\mathbb{C}^n$.

(e) **[+1]** A matrix $A \in \mathbb{C}^{n\times n}$ is called *unitary* if $A^{-1} = A^*$; i.e., $A^*A = I_n$. Show that $A \in \mathbb{C}^{n\times n}$ is unitary if and only if $\langle A\vec{x}, A\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

**Solution:** Again by part (c), we know that $\langle A\vec{x}, A\vec{y}\rangle = \langle \vec{x}, A^*A\vec{y}\rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$. Thus, by part (b), we see that $\langle A\vec{x}, A\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle = \langle \vec{x}, I_n\vec{y}\rangle$ for every $\vec{x}, \vec{y} \in \mathbb{C}^n$ if and only if $A^*A = I_n$. $\square$

(4) **Bonus[+1]** We will show in class that there are infinite-dimensional inner product spaces $V$ which have subspaces $S \leq V$ with $S^{\perp\perp} \neq S$. Despite this, prove that if $V$ is any inner product space and $S \leq V$, then $S^{\perp\perp\perp} = S^{\perp}$.

**Solution:** ($\supseteq$) For simplicity, set $T = S^{\perp}$. We want to show that $T^{\perp\perp} \supseteq T$, which we proved for a general $T$ in class.

($\subseteq$) Fix $x \in S^{\perp\perp\perp}$; we need to show that for any $s \in S$, we have $\langle x, s\rangle = 0$, so that $x \in S^{\perp}$. By definition of $S^{\perp\perp\perp}$, we know that for any $t \in S^{\perp\perp}$ we have $\langle x, t\rangle = 0$. However, $S \subseteq S^{\perp\perp}$, so we know that for any $s \in S$, we must have $\langle x, s\rangle = 0$ as needed. $\square$