Show your work and justify all answers.

(11 pts)

(1) This exercise will walk through a proof that not every norm is associated with an inner product.

  (a) **[+1]** Let V be an inner product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and let $\|\cdot\|$ be the associated norm. Prove that for any $x, y \in V$, we have

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big).$$

  (This is known as the parallelogram rule)

  (b) **[+1]** For a vector $\vec{x} \in \mathbb{C}^n$, the ℓ_1-norm of $\vec{x}$ is defined as $\|\vec{x}\|_1 = \sum_{k=1}^n |x_k|$. Prove that $\|\cdot\|_1$ is indeed a norm on $\mathbb{C}^n$.

  (c) **[+2]** Show that for $n \geq 2$, there is no inner product on $\mathbb{C}^n$ associated with the ℓ_1-norm.

(2) **[+2]** Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n with real coefficients as a vector space over $\mathbb{R}$. Equip $\mathcal{P}_n$ with the inner product $\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$. (While you do not have to prove it, convince yourself that $\langle \cdot, \cdot \rangle$ is indeed an inner product on $\mathcal{P}_n$)

  Use the Gram–Schmidt algorithm to find an orthonormal basis for $\mathcal{P}_2$ by starting with the basis $\{1, x, x^2\}$.

(3) Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^n$.

  (a) (Not graded. This is just a good fact to keep in mind.[1]) Fix $A, B \in \mathbb{C}^{m \times n}$. Show that $A = B$ if and only if $A\vec{x} = B\vec{x}$ for every $\vec{x} \in \mathbb{C}^n$.

  (b) **[+2]** Fix $A, B \in \mathbb{C}^{n \times n}$. Show that $A = B$ if and only if $\langle \vec{x}, A\vec{y} \rangle = \langle \vec{x}, B\vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

  (c) **[+1]** Show that if $A \in \mathbb{C}^{n \times n}$, then for any $\vec{x}, \vec{y} \in \mathbb{C}^n$, we have $\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle$.

  (d) **[+1]** A matrix $A \in \mathbb{C}^{n \times n}$ is called *Hermitian* if $A^* = A$. Show that $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if $\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A\vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

  (e) **[+1]** A matrix $A \in \mathbb{C}^{n \times n}$ is called *unitary* if $A^{-1} = A^*$; i.e., $A^*A = I_n$. Show that $A \in \mathbb{C}^{n \times n}$ is unitary if and only if $\langle A\vec{x}, A\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

(4) **Bonus[+1]** We will show in class that there are infinite-dimensional inner product spaces V which have subspaces $S \leq V$ with $S^{\perp\perp} \neq S$. Despite this, prove that if V is any inner product space and $S \leq V$, then $S^{\perp\perp\perp} = S^{\perp}$.

[1] wink wink nudge nudge