Show your work and justify all answers.

**11 pts**

(1) This exercise will walk through a proof that not every norm is associated with an inner product.

(a) [+] Let $V$ be an inner product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$ and let $\| \cdot \|$ be the associated norm. Prove that for any $x, y \in V$, we have

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2).$$

(This is known as the parallelogram rule)

(b) [+] For a vector $\vec{x} \in \mathbb{C}^n$, the $\ell_1$-norm of $\vec{x}$ is defined as $\| \vec{x} \|_1 = \sum_{k=1}^{n} |x_k|$. Prove that $\| \cdot \|_1$ is indeed a norm on $\mathbb{C}^n$.

(c) [+] Show that for $n \geq 2$, there is no inner product on $\mathbb{C}^n$ associated with the $\ell_1$-norm.

(2) [+] Let $P_n$ denote the space of polynomials of degree at most $n$ with real coefficients as a vector space over $\mathbb{R}$. Equip $P_n$ with the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx$. (While you do not have to prove it, convince yourself that $\langle \cdot, \cdot \rangle$ is indeed an inner product on $P_n$)

Use the Gram–Schmidt algorithm to find an orthonormal basis for $P_2$ by starting with the basis $\{1, x, x^2\}$.

(3) Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian inner product on $\mathbb{C}^n$.

(a) (Not graded. This is just a good fact to keep in mind) Fix $A, B \in \mathbb{C}^{m \times n}$. Show that $A = B$ if and only if $A \vec{x} = B \vec{x}$ for every $\vec{x} \in \mathbb{C}^n$.

(b) [+] Fix $A, B \in \mathbb{C}^{n \times n}$. Show that $A = B$ if and only if $\langle \vec{x}, A \vec{y} \rangle = \langle \vec{x}, B \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

(c) [+] Show that if $A \in \mathbb{C}^{n \times n}$, then for any $\vec{x}, \vec{y} \in \mathbb{C}^n$, we have $\langle A \vec{x}, \vec{y} \rangle = \langle \vec{x}, A^* \vec{y} \rangle$.

(d) [+] A matrix $A \in \mathbb{C}^{n \times n}$ is called Hermitian if $A^* = A$. Show that $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if $\langle A \vec{x}, \vec{y} \rangle = \langle \vec{x}, A^* \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

(e) [+] A matrix $A \in \mathbb{C}^{n \times n}$ is called unitary if $A^{-1} = A^*$; i.e., $A^* A = I_n$. Show that $A \in \mathbb{C}^{n \times n}$ is unitary if and only if $\langle A \vec{x}, A \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$.

(4) **Bonus** [+] We will show in class that there are infinite-dimensional inner product spaces $V$ which have subspaces $S \leq V$ with $S^\perp \neq S$. Despite this, prove that if $V$ is any inner product space and $S \leq V$, then $S^{\perp \perp} = S^\perp$.

---

1 wink wink nudge nudge