Show your work and justify all answers.

(9 pts)

(1) Fix $A, B \in \mathbb{R}^{m\times n}$.

(a) **[+1]** Show that $\operatorname{rank}(A+B) \le \operatorname{rank} A + \operatorname{rank} B$.

Solution: Let $\vec{b} \in \operatorname{Col}(A+B)$ so we can find $\vec{y} \in \mathbb{R}^n$ such that $(A+B)\vec{y} = \vec{b}$.
Now, $A\vec{y} \in \operatorname{Col} A$ and $B\vec{y} \in \operatorname{Col} B$, so $\vec{b}$ can be written as the sum of a vector in $\operatorname{Col} A$ and a vector in $\operatorname{Col} B$, so $\vec{b} \in \operatorname{Col} A + \operatorname{Col} B$.
Thus, $\operatorname{Col}(A+B) \subseteq \operatorname{Col} A + \operatorname{Col} B$, so the full claim follows from the fact that $\dim(S_1 + S_2) \le \dim S_1 + \dim S_2$, as was shown in problem (4) in DSW2. □

(b) **[+1]** Find an example of two non-zero matrices where $\operatorname{rank}(A+B) = \operatorname{rank} A + \operatorname{rank} B$.

Solution: Set $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Clearly $\operatorname{rank} A = \operatorname{rank} B = 1$ and $\operatorname{rank}(A+B) = 2$ since $A + B = I_2$. □

(2) Suppose that $A \in \mathbb{R}^{n\times n}$ has the property that $A^2 = A$.

(a) **[+1]** Show that $\operatorname{Col} A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{x}\}$.

Solution: ($\supseteq$) Certainly if $A\vec{x} = \vec{x}$, then $\vec{x} \in \operatorname{Col} A$.
($\subseteq$) Let $\vec{b} \in \operatorname{Col} A$, so we can find $\vec{y} \in \mathbb{R}^n$ for which $A\vec{y} = \vec{b}$. Therefore, $A\vec{b} = A^2\vec{y} = A\vec{y} = \vec{b}$, so $A\vec{b} = \vec{b}$ as needed. □

(b) **[+1]** Show that $\operatorname{Nul} A = \{\vec{x} \in \mathbb{R}^n : \vec{x} = \vec{u} - A\vec{u} \text{ for some } \vec{u} \in \mathbb{R}^n\}$.

Solution: ($\subseteq$) Suppose that $A\vec{x} = \vec{0}$ and set $\vec{u} = \vec{x}$. We then see that $\vec{x} = \vec{u} - \vec{0} = \vec{u} - A\vec{u}$ as needed.
($\supseteq$) If $\vec{x} = \vec{u} - A\vec{u}$ for some $\vec{u} \in \mathbb{R}^n$, then we see that $A\vec{x} = A\vec{u} - A^2\vec{u} = A\vec{u} - A\vec{u} = \vec{0}$. □

(c) **[+1]** Show that $\operatorname{Col} A \cap \operatorname{Nul} A = \{\vec{0}\}$.

Solution: Here are two solutions, the first being my favorite.

- Suppose that there were some nonzero $\vec{v} \in \operatorname{Col} A \cap \operatorname{Nul} A$. Since $\vec{v} \in \operatorname{Col} A$, there is some $\vec{u} \in \mathbb{R}^n$ such that $A\vec{u} = \vec{v}$. Thus, $A^2\vec{u} = A\vec{v} = \vec{0}$, so $\vec{u} \in \operatorname{Nul}(A^2)$ but since $\vec{v} \ne \vec{0}$, we know that $\vec{u} \notin \operatorname{Nul} A$; a contradiction since $A = A^2$.
- Certainly $\{\vec{0}\} \subseteq \operatorname{Col} A \cap \operatorname{Nul} A$, so we need only show the reverse containment. Suppose that $\vec{v} \in \operatorname{Col} A \cap \operatorname{Nul} A$; we need to show that $\vec{v} = \vec{0}$.
  Since $\vec{v} \in \operatorname{Nul} A$, we know that $A\vec{v} = \vec{0}$. But then by part (a), we know that $\vec{v} = A\vec{v} = \vec{0}$ since $\vec{v} \in \operatorname{Col} A$.
  Thus $\vec{v} = \vec{0}$.

□

(d) **[+2]** Show that $\operatorname{Col} A + \operatorname{Nul} A = \mathbb{R}^n$.

Solution: Here are two solutions, the first being my favorite.

- By part (c) and problem (5b) on DSW2, we see that $\dim\big(\operatorname{Col} A + \operatorname{Nul} A\big) = \dim \operatorname{Col} A + \dim \operatorname{Nul} A$. Furthermore, by the rank–nulity theorem, this tells us that $\dim\big(\operatorname{Col} A + \operatorname{Nul} A\big) = n$. Since certainly $\operatorname{Col} A + \operatorname{Nul} A \le \mathbb{R}^n$, this means that we must have $\operatorname{Col} A + \operatorname{Nul} A = \mathbb{R}^n$.

- Certainly $\operatorname{Col} A + \operatorname{Nul} A \subseteq \mathbb{R}^n$, so we need only show the reverse containment.

  Fix $\vec{u} \in \mathbb{R}^n$; we need to show that we can write $\vec{u} = \vec{c} + \vec{n}$ for some $\vec{c} \in \operatorname{Col} A$ and $\vec{n} \in \operatorname{Nul} A$. Set $\vec{n} = \vec{u} - A\vec{u}$, which is an element of $\operatorname{Nul} A$ by part (b). Now set $\vec{c} = \vec{u} - \vec{n} = A\vec{u}$, so $\vec{u} = \vec{c} + \vec{n}$. Of course, this means that $\vec{c} \in \operatorname{Col} A$ since there is a solution to $A\vec{x} = \vec{c}$.

$\square$

(3) **[+2]** Is there a matrix A for which $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \in \operatorname{Nul} A$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \in \operatorname{Col} A$? Why or why not?

Solution: No, there cannot be such an A. If such an A were to exist, then we know that $A \in \mathbb{R}^{3\times 3}$, so by the rank–nullity theorem, we would then have $\dim \operatorname{Nul} A + \dim \operatorname{Col} A = 3$. However, the conditions on A imply that $\dim \operatorname{Nul} A \geq 2$ and $\dim \operatorname{Col} A \geq 2$; an impossibility. $\square$

(4) **Bonus**[+1] Let $A, B \in \mathbb{R}^{m\times n}$. Show that if there is some non-zero $\vec{v} \in \mathbb{R}^m$ for which both $A\vec{x} = \vec{v}$ and $B\vec{x} = \vec{v}$ have a solution, then $\operatorname{rank}(A+B) < \operatorname{rank} A + \operatorname{rank} B$.

Solution: By the proof of problem (1), we know that $\operatorname{Col}(A+B) \subseteq \operatorname{Col} A + \operatorname{Col} B$. Problem (5d) on DSW2 implies that $\dim\big(\operatorname{Col} A + \operatorname{Col} B\big) = \dim \operatorname{Col} A + \dim \operatorname{Col} B - \dim\big(\operatorname{Col} A \cap \operatorname{Col} B\big)$, which proves the claim since $\operatorname{Col} A \cap \operatorname{Col} B$ is a non-trivial subspace by assumption since there is some non-zero $\vec{v} \in \operatorname{Col} A \cap \operatorname{Col} B$. $\square$

(5) Study for the midterm!