Show your work and justify all answers.

(9 pts)

(1) Fix $A, B \in \mathbb{R}^{m\times n}$.
   (a) [**+1**] Show that $\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$.
   (b) [**+1**] Find an example of two non-zero matrices where $\operatorname{rank}(A+B) = \operatorname{rank} A + \operatorname{rank} B$.

(2) Suppose that $A \in \mathbb{R}^{n\times n}$ has the property that $A^2 = A$.
   (a) [**+1**] Show that $\operatorname{Col} A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{x}\}$.
   (b) [**+1**] Show that $\operatorname{Nul} A = \{\vec{x} \in \mathbb{R}^n : \vec{x} = \vec{u} - A\vec{u} \text{ for some } \vec{u} \in \mathbb{R}^n\}$.
   (c) [**+1**] Show that $\operatorname{Col} A \cap \operatorname{Nul} A = \{\vec{0}\}$.
   (d) [**+2**] Show that $\operatorname{Col} A + \operatorname{Nul} A = \mathbb{R}^n$.

(3) [**+2**] Is there a matrix A for which $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \in \operatorname{Nul} A$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \in \operatorname{Col} A$? Why or why not?

(4) **Bonus**[+1] Let $A, B \in \mathbb{R}^{m\times n}$. Show that if there is some non-zero $\vec{v} \in \mathbb{R}^m$ for which both $A\vec{x} = \vec{v}$ and $B\vec{x} = \vec{v}$ have a solution, then $\operatorname{rank}(A+B) < \operatorname{rank} A + \operatorname{rank} B$.

(5) Study for the midterm!