Show your work and justify all answers.

(12 pts)

(1) **[+2]** Let $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$. For $x, y \in \mathbb{R}_{>0}$, define $x \oplus y = xy$ and for $c \in \mathbb{R}$, define $c \odot x = x^c$. Let $V = (\mathbb{R}_{>0}, \oplus, \odot)$ with $\oplus$ as vector addition and $\odot$ as scalar multiplication. Show that V is a vector space over $\mathbb{R}$. Be sure to verify all 10 axioms listed on page 78 of Hefferon. (You may take all of the basic properties of multiplication and exponentiation for granted)

Solution: Before we verify the 10 axioms, we need to know that $V \neq \varnothing$. But this is clear since certainly there are positive real numbers, such as 3435.

1. Certainly $x \oplus y = xy \in \mathbb{R}_{>0}$ for any $x, y \in \mathbb{R}_{>0}$ since the product of positive numbers is positive.
2. Since standard multiplication is commutative, so is $\oplus$.
3. Since standard multiplication is associative, so is $\oplus$.
4. We notice that $0_V = 1$ since $1 \oplus x = 1 \cdot x = x$ for any $x \in \mathbb{R}_{>0}$.
5. We notice that for any $\mathbb{R}_{>0}$, $\frac{1}{x} \in \mathbb{R}_{>0}$ and $\frac{1}{x} \oplus x = 1 = 0_V$, so every vector has an additive inverse.
6. Certainly for any $x \in \mathbb{R}_{>0}$ and $c \in \mathbb{R}$, we have $c \oplus x = x^c \in \mathbb{R}_{>0}$.
7. We verify for $c, d \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$, $(c + d) \oplus x = x^{c+d} = x^c x^d = (c \odot x) \oplus (d \odot x)$.
8. We verify for $c \in \mathbb{R}$ and $x, y \in \mathbb{R}_{>0}$, $c \odot (x \oplus y) = (xy)^c = x^c y^c = (c \odot x) \oplus (c \odot y)$.
9. We verify for $c, d \in \mathbb{R}$ and $x \in \mathbb{R}_{>0}$, $(cd) \odot x = x^{cd} = (x^d)^c = c \odot (d \odot x)$.
10. For any $x \in \mathbb{R}_{>0}$, we have $1 \odot x = x^1 = x$.

□

(2) **[+1]** Let $V = (\mathbb{R}_+, \oplus, \odot)$ where $\oplus$ and $\odot$ are as in problem (1) and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Is this new V still a vector space?

Solution: No, axiom 5 fails to hold. Notice that $0_V = 1$ still. However, $0 \in \mathbb{R}_+$, but there is no $x \in \mathbb{R}_+$ with $x \oplus 0 = 1 = 0_V$, i.e. 0 does not have an additive inverse. □

(3) **[+1]** Fix $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Define $E = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \lambda\vec{x}\}$. Prove that E is a subspace of $\mathbb{R}^n$ (this is known as the λ-eigenspace of A).

Solution: Firstly, $A\vec{0} = \vec{0} = \lambda\vec{0}$, so $\vec{0} \in E$. Now, if $\vec{x}, \vec{y} \in E$ and $c, d \in \mathbb{R}$, then $A(c\vec{x}+d\vec{y}) = cA\vec{x}+dA\vec{y} = c\lambda\vec{x} + d\lambda\vec{y} = \lambda(c\vec{x} + d\vec{y})$. Therefore, $c\vec{x} + d\vec{y} \in E$. We conclude that E is a subspace of $\mathbb{R}^n$.

Alternatively, we could notice that $A\vec{x} = \lambda\vec{x}$ if and only if $(A - \lambda I_n)\vec{x} = 0$, so if $B = A - \lambda I_n$, then $E = \{\vec{x} \in \mathbb{R}^n : B\vec{x} = \vec{0}\}$. We proved in class that any set of this form is a subspace. □

(4) **[+2]** Use induction to prove that if $A_1, \ldots, A_n \in \mathbb{R}^{m \times m}$ are non-singular matrices, then their product $A_1 \cdots A_n$ is non-singular as well. You may freely use the results of any problems on previous homeworks or discussion sessions. (Beware of the "all horses are brown" trap!)

Solution: We prove by induction on n.

Base Cases: $n = 1$ is trivial since A_1 is non-singular whenever A_1 is non-singular.

For $n = 2$, problem (5c) from DSW1 shows that if A_1, A_2 are non-singular, then so is $A_1 A_2$.

Induction hypothesis: For some $N > 2$, if $A_1, \ldots, A_{N-1} \in \mathbb{R}^{m \times m}$ are non-singular, then so is $A_1 \cdots A_{N-1}$.

Induction step: Let $A_1, \dots, A_N \in \mathbb{R}^{m\times m}$ be non-singular. Assuming the induction hypothesis, we need to show that $A_1 \cdots A_N$ is also non-singular.

Firstly, set $B = A_1 \cdots A_{N-1}$, so $A_1 \cdots A_N = BA_N$. Now, by the induction hypothesis, we know that B is non-singular. Since also A_N is non-singular by assumption, we can apply the $n = 2$ case (which was proved as a base case) to conclude that BA_N is non-singular as needed. □

(5) **[+3]** Let V be a vector space and let $U, W \leq V$ (recall that "$\leq$" here means "is a subspace of"). Prove that $U \cup W$ is a subspace if and only if either $U \subseteq W$ or $W \subseteq U$.

Solution: ($\Leftarrow$) Without loss of generality, suppose $U \subseteq W$ (the other case is symmetric). Then $U \cup W = W$, which we know is a subspace by assumption.

($\Rightarrow$) We prove the contrapositive. Suppose that $U \not\subseteq W$ and $W \not\subseteq U$. This means that there is some $u \in U \setminus W$ and $w \in W \setminus U$. Note that $u, w \in U \cup W$; we claim that $u + w \notin U \cup W$, thus showing that $U \cup W$ is not a subspace.

Suppose for the sake of contradiction that $u + w \in U \cup W$, so either $u + w \in U$ or $u + w \in W$. Without loss of generality, suppose that $u + w \in U$ (the other case is symmetric). Since $u \in U$ and U is a subspace, this means that we must have $w = (u + w) - u \in U$; a contradiction. □

(6) Let V be a vector space and $S, T \subseteq V$ (not necessarily subspaces).

(a) **[+1]** Must it be the case that $\operatorname{span}(S \cup T) = \operatorname{span} S \cup \operatorname{span} T$?

Solution: No. Let, say, $V = \mathbb{R}^2$ and let S, T be any subspaces of V with $S \not\subseteq T$ and $T \not\subseteq S$, e.g. $T = \operatorname{span} \vec{e}_1$ and $S = \operatorname{span} \vec{e}_2$. We know that $\operatorname{span}(S \cup T)$ is a subspace of V, yet by problem (5), we know that $\operatorname{span} S \cup \operatorname{span} T = S \cup T$ is not a subspace, so they cannot be equal. □

(b) **[+1]** Must it be the case that $\operatorname{span}(S \cap T) = \operatorname{span} S \cap \operatorname{span} T$?

Solution: No. Consider $V = \mathbb{R}$, $S = \{\vec{e}_1\}$ and $T = \{2\vec{e}_1\}$. Notice that $\operatorname{span} S = \operatorname{span} T = \mathbb{R}$, so $\operatorname{span} S \cap \operatorname{span} T = \mathbb{R}$. However, $\operatorname{span}(S \cap T) = \operatorname{span} \varnothing = \{\vec{0}\}$. □

(7) **[+1]** Find a set of three vectors $\{v_1, v_2, v_3\}$ which is linearly *dependent*, but $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are all linearly *independent*. (You get to pick the vector space)

Solution: Say our vector space is $\mathbb{R}^2$. Pick $v_1 = \vec{e}_1$, $v_2 = \vec{e}_2$ and $v_3 = \vec{1}$ where $\vec{1}$ is the 2-dimensional all-ones vector. We see that $\{v_1, v_2, v_3\}$ is linearly dependent since $\vec{e}_1 + \vec{e}_2 - \vec{1} = \vec{0}$.

For $\{\vec{e}_1, \vec{e}_2\}$, if we had $\vec{0} = c_1\vec{e}_1 + c_2\vec{e}_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then certainly $c_1 = c_2 = 0$, so these vectors are linearly independent.

For $\{\vec{e}_1, \vec{1}\}$, if we had $\vec{0} = c_1\vec{e}_1 + c_2\vec{1} = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$, then solving this linear system tells us that $c_1 = c_2 = 0$, so these vectors are also linearly independent.

The same reasoning holds to show that $\{\vec{e}_2, \vec{1}\}$ is linearly independent. □