Show your work and justify all answers.

(12 pts)

(1) **[+2]** Let $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$. For $x, y \in \mathbb{R}_{>0}$, define $x \oplus y = xy$ and for $c \in \mathbb{R}$, define $c \odot x = x^c$. Let $V = (\mathbb{R}_{>0}, \oplus, \odot)$ with $\oplus$ as vector addition and $\odot$ as scalar multiplication. Show that V is a vector space over $\mathbb{R}$. Be sure to verify all 10 axioms listed on page 78 of Hefferon. (You may take all of the basic properties of multiplication and exponentiation for granted)

(2) **[+1]** Let $V = (\mathbb{R}_+, \oplus, \odot)$ where $\oplus$ and $\odot$ are as in problem (1) and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Is this new V still a vector space?

(3) **[+1]** Fix $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Define $E = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \lambda\vec{x}\}$. Prove that E is a subspace of $\mathbb{R}^n$ (this is known as the λ-eigenspace of A).

(4) **[+2]** Use induction to prove that if $A_1, \ldots, A_n \in \mathbb{R}^{m \times m}$ are non-singular matrices, then their product $A_1 \cdots A_n$ is non-singular as well. You may freely use the results of any problems on previous homeworks or discussion sessions. (Beware of the "all horses are brown" trap!)

(5) **[+3]** Let V be a vector space and let $U, W \leq V$ (recall that "$\leq$" here means "is a subspace of"). Prove that $U \cup W$ is a subspace if and only if either $U \subseteq W$ or $W \subseteq U$.

(6) Let V be a vector space and $S, T \subseteq V$ (not necessarily subspaces).
  (a) **[+1]** Must it be the case that $\operatorname{span}(S \cup T) = \operatorname{span} S \cup \operatorname{span} T$?
  (b) **[+1]** Must it be the case that $\operatorname{span}(S \cap T) = \operatorname{span} S \cap \operatorname{span} T$?

(7) **[+1]** Find a set of three vectors $\{v_1, v_2, v_3\}$ which is linearly *dependent*, but $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are all linearly *independent*. (You get to pick the vector space)