Show your work and justify all answers.

(11 pts)

(1) [**+2**] Find $\begin{bmatrix} 0 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1}$ by using Gaussian elimination.

Solution:

$$\left[\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{r_1 \to r_3 \to r_2 \to r_1}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 3 & -1 & 1 & 0 & 0 \end{array}\right] \overset{r_2 - r_1,\ (-1)\cdot r_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 3 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\overset{r_3 - 3r_2,\ (-\frac{1}{4})\cdot r_3}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} \end{array}\right] \overset{r_1 - r_3,\ r_2 - r_3}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} \end{array}\right]$$

Therefore, $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ -1 & 3 & -3 \end{bmatrix}$. □

(2) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ are arbitrary.

(a) [**+1**] Show that $AB = (ad - bc)I_2$.

Solution: $AB = \begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & ad - bc \end{bmatrix} = (ad - bc)I_2$. □

(b) [**+2**] Show that A is non-singular if and only if $ad - bc \neq 0$.
(You may use the result of any other problem on this assignment, even if you haven't proved it)

Solution: If $ad - bc \neq 0$, then part (a) shows us that $A^{-1} = \frac{1}{ad-bc}B$, so A is non-singular.
On the other hand, if $ad - bc = 0$, then $AB = O_2$. If it were to be the case that A was non-singular, then by problem (4), we know that it must be the case that $B = O_2$. However, if $B = O_2$, then $a = b = c = d = 0$, so $A = O_2$ as well; an impossibility. Hence, A cannot be non-singular. □

(3) [**+1**] Suppose that $A, B \in \mathbb{R}^{n \times n}$ are both non-singular. Must $A + B$ also be non-singular?

Solution: No. Let A be any non-singular matrix and set $B = -A$. Certainly B is also non-singular, but $A + B = A - A = O_n$, which is singular. □

(4) [**+2**] Let $A, B \in \mathbb{R}^{n \times n}$ where A is non-singular. Prove that $AB = O_n$ if and only if $B = O_n$. Here O_n is the $n \times n$ zero matrix.

Solution: Certainly if $B = O_n$, then $AB = O_n$ since O_n is the zero-matrix.
For the other direction, here are two ways to do it:

- Since A is non-singular, A^{-1} exists, so multiply both sides of $AB = O_n$ by A^{-1} on the left, so $A^{-1}AB = A^{-1}O_n \implies B = O_n$ since $A^{-1}A = I_n$ and $A^{-1}O_n = O_n$.

- Start by writing $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$, so

$$O_n = AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{bmatrix}.$$

  Therefore, $A\vec{b}_i = \vec{0}$ for all i. Since A is non-singular, we know that the only solution to $A\vec{x} = \vec{0}$ is the trivial solution, so $\vec{b}_i = \vec{0}$ for all i. In other words, $B = O_n$.

$\square$

(5) **[+3]** For a matrix $A \in \mathbb{R}^{m\times n}$, the ith row-sum of A is $\sum_{j=1}^{n} A_{ij}$ (i.e. the sum of the entries in the ith row).

Let $A \in \mathbb{R}^{n\times n}$ be non-singular, and suppose that every row-sum of A is equal to k. What are the row-sums of A^{-1}?

Solution: The row-sums of A^{-1} are all $1/k$.

To see this, let $\vec{1}$ denote the all-one vector, then the vector of row-sums of a matrix B is precisely $B\vec{1}$.

By assumption, we know that $A\vec{1} = k\vec{1}$ since all row-sums of A are k. Notice that this means that $k \neq 0$, otherwise $A\vec{x} = \vec{0}$ will have a non-trivial solution. Multiplying on the left by A^{-1} yields $\vec{1} = kA^{-1}\vec{1}$, so $A^{-1}\vec{1} = \frac{1}{k}\vec{1}$, so the row-sums of A^{-1} are all $1/k$. $\square$