Show your work and justify all answers.

(9 pts)

(1) [**+2**] Solve for $\vec{x}$ in the following linear system by finding a particular solution and the homogeneous solution(s). Write your answer in vector form, e.g. $\{\vec{u} + t\vec{v} + s\vec{w} : t, s \in \mathbb{R}\}$.

$$\begin{bmatrix} 2 & 4 & 0 \\ 2 & -2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Solution: We begin by finding a particular solution:

$$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 2 & -2 & 1 & 0 \end{array}\right] \overset{r_2 - r_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 0 & -6 & 1 & -2 \end{array}\right] \overset{r_1/2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & -6 & 1 & -2 \end{array}\right]$$

A particular solution is then found by picking, say $x_2 = 0$, which gives $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ as a particular solution.

We can then use the same reductions to find the homogeneous solution(s): we get the equations $x_1 + 2x_2 = 0$ and $-6x_2 + x_3 = 0$. Choosing, say, x_2 to be our free variable, we see that the solutions to $A\vec{x} = \vec{0}$ are of the form $\begin{bmatrix} -2x_2 \\ x_2 \\ 6x_2 \end{bmatrix}$ for any $x_2 \in \mathbb{R}$. Therefore, the solutions to $A\vec{x} = \vec{b}$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

□

(2) [**+2**] With $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$, is there any $\vec{b} \in \mathbb{R}^3$ for which $A\vec{x} = \vec{b}$ has infinitely many solutions for $\vec{x}$?

Solution: Begin by row-reducing A to echelon form:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \overset{r_2 - r_1}{\rightsquigarrow} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 2 & 0 \end{bmatrix} \overset{r_3 - 2r_1}{\rightsquigarrow} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & -2 \end{bmatrix} \overset{r_3 - r_2}{\rightsquigarrow} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}.$$

We see that there are two pivots, so rank $A = 2$. Since A has 2 columns, A has full-column rank, so we know that there is at most one solution to the equation $A\vec{x} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^3$. □

(3) [**+2**] With the same A as in problem (2), is there any $\vec{b} \in \mathbb{R}^3$ for which $A\vec{x} = \vec{b}$ does not have any solution?

Solution: By problem (2), we know that A does not have full-row rank, so there must be some $\vec{b} \in \mathbb{R}^3$ for which $A\vec{x} = \vec{b}$ has no solution. □

(4) **[+3]** Recall that for $A \in \mathbb{R}^{m\times n}$, a matrix $R \in \mathbb{R}^{n\times m}$ is called a right-inverse of A if $AR = I_m$ and a matrix $L \in \mathbb{R}^{n\times m}$ is called a left-inverse if $LA = I_n$.

Show that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ has infinitely many left-inverses, but does not have a right-inverse.

Solution: For $x \in \mathbb{R}$, let $L_x = \begin{bmatrix} x & x-1 & 1-x \\ x-1 & x & 1-x \end{bmatrix}$. By direct computation, we find that $L_x A = I_2$ for any $x \in \mathbb{R}$, so A has infinitely many left-inverses.

Now, if A were to have a right-inverse, $R \in \mathbb{R}^{2\times 3}$, then for any $\vec{b} \in \mathbb{R}^3$, there would be a solution to $A\vec{x} = \vec{b}$, namely $\vec{x} = R\vec{b}$. However, clearly $\operatorname{rank} A = 2$, but A has three rows, meaning that there must be some $\vec{b} \in \mathbb{R}^3$ for which $A\vec{x} = \vec{b}$ does not have a solution; a contradiction. Therefore, A cannot have a right-inverse. $\square$

(5) **Bonus[+1]** Show that a non-square matrix $A \in \mathbb{R}^{m\times n}$ (that is, with $m \neq n$) cannot have both a left-inverse and a right-inverse.

Solution: First suppose that $m > n$, i.e. A has more rows than columns. We know that $\operatorname{rank} A \leq \min\{m, n\}$, so if $m > n$, we know that A does not have full row rank. Using the same argument as in problem (4), we know that this means that A cannot have a right-inverse.

If $m < n$, consider instead A^T and suppose for the sake of contradiction that A were to have a left-inverse L. We would then have $I_n = I_n^T = (LA)^T = A^T L^T$, so L^T is a right-inverse of A^T. However, $A^T \in \mathbb{R}^{n\times m}$, so A^T has more rows than columns. But we just proved above that such a matrix cannot have a right-inverse; a contradiction. $\square$