Show your work and justify all answers.

(11 pts)

(1) **[+2]** Show that $A \in \mathbb{C}^{n\times n}$ is diagonalizable and has real eigenvalues if and only if $A = BC$ where $B \succ 0$ and C is Hermitian.

Solution: ($\Leftarrow$) We essentially showed this in class. Since $B \succ 0$, we can write $A = B^{1/2}(B^{1/2}CB^{1/2})B^{-1/2}$, so A is similar to $B^{1/2}CB^{1/2}$. Now, since C is Hermitian, $B^{1/2}CB^{1/2}$ is Hermitian as well. We conclude that A is diagonalizable and has real eigenvalues since the same is true of $B^{1/2}CB^{1/2}$.

($\Rightarrow$) Since A is diagonalizable and has real eigenvalues, we can write $A = PDP^{-1}$ where $D \in \mathbb{R}^{n\times n}$ is diagonal. Then $A = P\big(P^*(P^*)^{-1}\big)DP^{-1} = (PP^*)\big((P^*)^{-1}DP^{-1}\big)$. Set $B = PP^*$ and $C = (P^*)^{-1}DP^{-1}$, so $A = BC$. Now, $B = PP^*$ so $B \succ 0$ since P is non-singular. Also, $C^* = C$ since $D = D^*$. □

(2) **[+1]** Show that if $A \in \mathbb{C}^{n\times n}$ is diagonalizable, then $\operatorname{Nul}(A^2) = \operatorname{Nul} A$.

Solution: Certainly $\operatorname{Nul} A \subseteq \operatorname{Nul}(A^2)$, so it suffices to show that $\dim \operatorname{Nul} A = \dim \operatorname{Nul}(A^2)$. Now, $\operatorname{Nul} A = E_0(A)$ and $\operatorname{Nul}(A^2) = E_0(A^2)$. Since A is diagonalizable, we know that if 0 has algebraic multiplicity k as an eigenvector for A (which could be $k = 0$), then $\dim E_0(A) = k$ as well. Now, $0^2 = 0$ is an eigenvalue for A^2 also with algebraic multiplicity k, so $k = \dim E_0(A) \le \dim E_0(A^2) \le k$. Thus, $\dim \operatorname{Nul}(A) = \dim \operatorname{Nul}(A^2)$, so we conclude that $\operatorname{Nul} A = \operatorname{Nul}(A^2)$.

Alternatively, we can use our classification of diagonalizable matrices attained through the proof of Jordan's theorem: A is diagonalizable if and only if $E_\lambda(A) = E_\lambda^2(A)$ for every $\lambda \in \mathbb{C}$. To this end, we simply note that $E_0(A) = \operatorname{Nul} A$ and $E_0^2(A) = \operatorname{Nul}(A^2)$, so the claim follows immediately. □

(3) **[+1]** Fix $A \in \mathbb{C}^{n\times n}$. Show that if $E_\lambda^k(A) = E_\lambda^{k+1}(A)$, for some $k \ge 1$, then $E_\lambda^t(A) = E_\lambda^k(A)$ for all $t \ge k$.

Solution: We prove this by induction on t.

Base case: $t = k$ is trivially true and $t = k+1$ is true by assumption.

Induction hypothesis: For some $t > k+1$, we have $E_\lambda^{t-1}(A) = E_\lambda^k(A)$.

Induction step: We know that $E_\lambda^k(A) \subseteq E_\lambda^t(A)$ since $t \ge k$, so we need only prove the reverse containment.

Fix $\vec{x} \in E_\lambda^t(A)$, so $\vec{0} = (A - \lambda I_n)^t\vec{x} = (A - \lambda I_n)^{t-1}(A - \lambda I_n)\vec{x}$, so $(A - \lambda I_n)\vec{x} \in E_\lambda^{t-1}(A) = E_\lambda^k(A)$, by the induction hypothesis, so $\vec{0} = (A - \lambda I_n)^k(A - \lambda I_n)\vec{x} = (A - \lambda I_n)^{k+1}\vec{x}$, so $\vec{x} \in E_\lambda^{k+1}(A)$. Since $E_\lambda^{k+1}(A) = E_\lambda^k(A)$ by assumption, this means that $\vec{x} \in E_\lambda^k(A)$, so $E_\lambda^t(A) \subseteq E_\lambda^k(A)$. □

(4) **[+2]** Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Find matrices P, J such that J is a Jordan form and $A = PJP^{-1}$.

Solution: We first notice that the eigenvalues of A are 1 with multiplicity 3. Now,

$$E_1(A) = \operatorname{Nul}(A - I_3) = \operatorname{Nul} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{\vec{e}_1\}.$$

Hence, 1 has geometric multiplicity 1, so we need to look at generalized eigenspaces.

$$E_1^2(A) = \operatorname{Nul}\big((A - I_3)^2\big) = \operatorname{Nul}\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\{\vec{e}_1, \vec{e}_2\}.$$

$E_1^2(A)$ still does not have dimension 3, so we need to compute

$$E_1^3(A) = \operatorname{Nul}\big((A - I_3)^3\big) = \operatorname{Nul} O_3 = \mathbb{C}^3,$$

which finally has dimension 3.

Now, to find the appropriate generalized eigenvectors, we begin by selecting $\vec{v}_3 \in E_1^3(A) \setminus E_1^2(A)$, so we can pick $\vec{v}_3 = \vec{e}_3$. Now, $\vec{v}_2 = (A - I_3)\vec{v}_3 = 2\vec{e}_2$, and then $\vec{v}_1 = (A - I_3)\vec{v}_2 = 2\vec{e}_1$.

Thus, with $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, we have $A = PJP^{-1}$. □

(5) Fix $A, B \in \mathbb{C}^{n \times n}$. We say that A and B are *simultaneously diagonalizable* if there is a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{C}^n$ such that each $\vec{v}_i$ is an eigenvector for both A and B.

(a) **[+1]** Show that if A and B are simultaneously diagonalizable, then $AB = BA$.

Solution: Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be this common basis of eigenvectors and suppose that $A\vec{v}_i = \lambda_i \vec{v}_i$ and $B\vec{v}_i = \mu_i \vec{v}_i$. Then $AB\vec{v}_i = \mu A\vec{v}_i = \mu\lambda\vec{v}_i = \lambda B\vec{v}_i = BA\vec{v}_i$, so $AB = BA$ by the linear extension lemma. □

(b) **[+1]** Suppose that A has (distinct) eigenvalues $\lambda_1, \ldots, \lambda_k$ and B has (distinct) eigenvalues $\mu_1, \ldots, \mu_\ell$ (but these eigenvalues may have higher multiplicity). Show that if

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} \big(E_{\lambda_i}(A) \cap E_{\mu_j}(B)\big) = \mathbb{C}^n.$$

then A, B are simultaneously diagonalizable.

Solution: By assumption, there is a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{C}^n$ such that, for each $k \in [n]$, $\vec{v}_k \in E_{\lambda_i}(A) \cap E_{\mu_j}(B)$ for some $i \in [k]$ and $j \in [\ell]$. In other words, $\vec{v}_k$ is an eigenvector for both A and B, so A and B are simultaneously diagonalizable. □

(c) **[+1]** Show that if $AB = BA$, then $E_\lambda(A)$ is a B-invariant subspace.

Solution: Let $\vec{x} \in E_\lambda(A)$; we need to show that $B\vec{x} \in E_\lambda(A)$ as well. Indeed, $A(B\vec{x}) = B(A\vec{x}) = \lambda B\vec{x}$, so $B\vec{x} \in E_\lambda(A)$. □

(d) **[+2]** Show that if A and B are both diagonalizable and $AB = BA$, then A and B are simultaneously diagonalizable.

Solution: We showed in class that if B is diagonalizable and $U \leq \mathbb{C}^n$ is a B-invariant subspace, then $U = \sum_{j=1}^{\ell}(U \cap E_{\mu_j}(B))$. Since $E_\lambda(A)$ is B-invariant, this means that for every $i \in [k]$, we have $E_{\lambda_i}(A) = \sum_{j=1}^{\ell}(E_{\lambda_i}(A) \cap E_{\mu_j}(B))$. Finally, since A is diagonalizable, we have

$$\mathbb{C}^n = \sum_{i=1}^{k} E_{\lambda_i}(A) = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \big(E_{\lambda_i}(A) \cap E_{\mu_j}(B)\big),$$

so A and B are simultaneously diagonalizable by part (b). □