Show your work and justify all answers.

(11 pts)

(1) [+**2**] Show that $A \in \mathbb{C}^{n\times n}$ is diagonalizable and has real eigenvalues if and only if $A = BC$ where $B \succ 0$ and C is Hermitian.

(2) [+**1**] Show that if $A \in \mathbb{C}^{n\times n}$ is diagonalizable, then $\operatorname{Nul}(A^2) = \operatorname{Nul} A$.

(3) [+**1**] Fix $A \in \mathbb{C}^{n\times n}$. Show that if $E_\lambda^k(A) = E_\lambda^{k+1}(A)$, for some $k \geq 1$, then $E_\lambda^t(A) = E_\lambda^k(A)$ for all $t \geq k$.

(4) [+**2**] Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Find matrices P, J such that J is a Jordan form and $A = PJP^{-1}$.

(5) Fix $A, B \in \mathbb{C}^{n\times n}$. We say that A and B are *simultaneously diagonalizable* if there is a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{C}^n$ such that each $\vec{v}_i$ is an eigenvector for both A and B.

(a) [+**1**] Show that if A and B are simultaneously diagonalizable, then $AB = BA$.

(b) [+**1**] Suppose that A has (distinct) eigenvalues $\lambda_1, \dots, \lambda_k$ and B has (distinct) eigenvalues $\mu_1, \dots, \mu_\ell$ (but these eigenvalues may have higher multiplicity). Show that if

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}\big(E_{\lambda_i}(A) \cap E_{\mu_j}(B)\big) = \mathbb{C}^n.$$

then A, B are simultaneously diagonalizable.

(c) [+**1**] Show that if $AB = BA$, then $E_\lambda(A)$ is a B-invariant subspace.

(d) [+**2**] Show that if A and B are both diagonalizable and $AB = BA$, then A and B are simultaneously diagonalizable.