Show your work and justify all answers.

(11 pts)

(1) **[+1]** Suppose that A is a non-singular matrix with $A, A^{-1} \in \mathbb{Z}^{n\times n}$; that is both A and A^{-1} have only integer entries. What are the possible values for $\det A$?

Solution: Since $\det A = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \prod_{i=1}^n A_{i,\sigma(i)}$ and $A_{i,j}$ is an integer for all i, j, we observe that $\det A$ must be an integer. The same reasoning tells us that $\det(A^{-1})$ must be an integer. Since $\det(A^{-1}) = \frac{1}{\det A}$, this means that both $\det A$ and $\frac{1}{\det A}$ are integers, meaning that $\det A = \pm 1$. □

(2) **[+1]** Show that if n is odd, then $A - A^T$ is singular for all $A \in \mathbb{C}^{n\times n}$.

Solution: We compute

$$\det(A - A^T) = \det\big((A - A^T)^T\big) = \det(A^T - A) = \det\big(-(A - A^T)\big) = (-1)^n \det(A - A^T) = -\det(A - A^T),$$

since n is odd. Hence, $\det(A - A^T) = 0$, so $A - A^T$ is singular. □

(3) **[+1]** Find the eigenvalues and eigenspaces of $\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$.

Solution: We first compute

$$P_A(t) = \det \begin{bmatrix} t-5 & 3 \\ -6 & t+4 \end{bmatrix} = (t-5)(t+4) + 18 = t^2 - t - 2 = (t-2)(t+1).$$

Therefore, the eigenvalues of A are 2 and -1, each with multiplicity one. From this, we compute

$$E_2(A) = \operatorname{Nul}\begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$E_{-1}(A) = \operatorname{Nul}\begin{bmatrix} -6 & 3 \\ -6 & 3 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

□

(4) **[+2]** Suppose that $A \in \mathbb{C}^{n\times n}$ satisfies $A^2 = A$. Prove that A is diagonalizable.
(Hint: you may find some inspiration in a previous homework)

Solution: We use the results of problem (2) in HW5 (notice that we never used the fact that A is real in the solution there, so the fact that A may be complex here is unimportant). We have $E_1(A) = \{\vec{x} \in \mathbb{C}^n : A\vec{x} = \vec{x}\} = \operatorname{Col} A$ and $E_0(A) = \{\vec{x} \in \mathbb{C}^n : A\vec{x} = \vec{0}\} = \operatorname{Nul} A$. Then we know that $E_1(A) + E_0(A) = \operatorname{Col} A + \operatorname{Nul} A = \mathbb{C}^n$, so A has a basis of eigenvectors, so A is diagonalizable. □

(5) Suppose that $A \in \mathbb{C}^{n\times n}$ has characteristic polynomial $P_A(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$ where $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ are distinct; that is, A has eigenvalues $\lambda_1, \dots, \lambda_k$ where λ_i has multiplicity m_i. Define $Q_A(t) = (t-\lambda_1)\cdots(t-\lambda_k)$.

(a) **[+1]** Show that if A is diagonalizable, then $Q_A(A) = O_n$.

Solution: Since A is diagonalizable, we know that there is a basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{C}^n$ such that each $\vec{v}_i$ is an eigenvector for A. Fix $i \in [n]$ and find $j \in [k]$ for which $A\vec{v}_i = \lambda_j \vec{v}_i$, so that $(A - \lambda_j I_n)\vec{v}_i = \vec{0}$.

Therefore, since powers of A commute,

$$Q_A(A)\vec{v}_i = (A-\lambda_1 I_n)\cdots(A-\lambda_k I_n) = \left(\prod_{\ell\neq j}(A-\lambda_\ell I_n)\right)(A-\lambda_j I_n)\vec{v}_i = \left(\prod_{\ell\neq j}(A-\lambda_\ell I_n)\right)\vec{0} = \vec{0}.$$

We conclude that $Q_A(A) = O_n$ by the linear extension lemma. □

(b) **[+1]** Find an example of some $A \in \mathbb{C}^{n\times n}$ for which $Q_A(A) \neq O_n$.

Solution: Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so $P_A(t) = \det \begin{bmatrix} t & -1 \\ 0 & t \end{bmatrix} = t^2$ and $Q_A(t) = t$, so $Q_A(A) \neq O_n$. □

(6) **[+2]** Show that $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if there is an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{C}^n$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $A = \sum_{i=1}^n \lambda_i \vec{v}_i \vec{v}_i^*$.

Solution: If $A = \sum_{i=1}^n \lambda_i \vec{v}\vec{v}^*$, then $A^* = \sum_{i=1}^n \lambda_i^* (\vec{v}_i\vec{v}_i^*)^* = \sum_{i=1}^n \lambda_i \vec{v}_i\vec{v}_i^* = A$ since $\lambda_i \in \mathbb{R}$.

On the other hand, if A is Hermitian, then the spectral theorem tells us that there is an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{C}^n$ such that each $\vec{v}_i$ is an eigenvector with $A\vec{v}_i = \lambda_i \vec{v}_i$ where $\lambda_i \in \mathbb{R}$. We claim that $A = \sum_{i=1}^n \lambda_i \vec{v}_i \vec{v}_i^*$.

Indeed, since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis, we have $\left(\sum_{i=1}^n \lambda_i \vec{v}_i\vec{v}_i^*\right)\vec{v}_k = \lambda_k \vec{v}_k \vec{v}_k^* \vec{v}_k = \lambda_k \vec{v}_k = A\vec{v}_k$. Thus, these matrices are equal through the linear extension lemma. □

(7) For a matrix $A \in \mathbb{C}^{n\times n}$, the *trace* of A is defined as $\operatorname{tr} A = \sum_{i=1}^n A_{ii}$; that is, the sum of the diagonal entries.

(a) **[+1]** Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Solution: We simply compute

$$\operatorname{tr}(AB) = \sum_{i=1}^m (AB)_{ii} = \sum_{i=1}^m \sum_{j=1}^n A_{ij}B_{ji} = \sum_{j=1}^n \sum_{i=1}^m B_{ji}A_{ij} = \sum_{j=1}^n (BA)_{jj} = \operatorname{tr}(BA).$$

□

(b) **[+1]** Let $A \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (not necessarily distinct). Show that $\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$.

Solution: We can write $A = UTU^*$ where U is unitary and T is upper-triangular. We know that the eigenvalues of T are also $\lambda_1, \dots, \lambda_n$ since A and T are similar. Additionally, since T is upper-triangular, its eigenvalues are precisely its diagonal entries; thus $\operatorname{tr} T = \lambda_1 + \cdots + \lambda_n$. Now, by part (a), $\operatorname{tr} A = \operatorname{tr}(UTU^*) = \operatorname{tr}(TUU^*) = \operatorname{tr}(TI_n) = \operatorname{tr} T = \lambda_1 + \cdots + \lambda_n$. □

(8) **Bonus[+1]** Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be *any* orthonormal basis for $\mathbb{C}^n$. Show that $\operatorname{tr} A = \sum_{k=1}^n \langle \vec{v}_i, A\vec{v}_i \rangle$ where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product.

Solution: We first show that if $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{C}^n$, then $\sum_{i=1}^n \vec{v}_i\vec{v}_i^* = I_n$. Indeed, $\left(\sum_{i=1}^n \vec{v}_i\vec{v}_i^*\right)\vec{v}_k = \vec{v}_k\vec{v}_k^*\vec{v}_k = \vec{v}_k$ since this is an orthonormal basis. Thus, $\sum_{i=1}^n \vec{v}_i\vec{v}_i^* = I_n$ by the linear extension lemma.

Now, $\langle \vec{v}_i, A\vec{v}_i \rangle = \operatorname{tr}\langle \vec{v}_i, A\vec{v}_i \rangle$ since this is simply a scalar, so we can compute using part (a) and the observation that $\operatorname{tr}(aA + bB) = a \operatorname{tr} A + b \operatorname{tr} B$ for scalars $a, b \in \mathbb{C}$ and $A, B \in \mathbb{C}^{n\times n}$,

$$\sum_{k=1}^n \langle \vec{v}_i, A\vec{v}_i \rangle = \sum_{k=1}^n \operatorname{tr}(\vec{v}_i^* A \vec{v}_i) = \sum_{k=1}^n \operatorname{tr}(A\vec{v}_i\vec{v}_i^*) = \operatorname{tr}\left(A \sum_{i=1}^n \vec{v}_i\vec{v}_i^*\right) = \operatorname{tr}(AI_n) = \operatorname{tr} A.$$

□