Show your work and justify all answers.

(11 pts)

(1) [**+1**] Suppose that A is a non-singular matrix with $A, A^{-1} \in \mathbb{Z}^{n\times n}$; that is both A and A^{-1} have only integer entries. What are the possible values for $\det A$?

(2) [**+1**] Show that if n is odd, then $A - A^T$ is singular for all $A \in \mathbb{C}^{n\times n}$.

(3) [**+1**] Find the eigenvalues and eigenspaces of $\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$.

(4) [**+2**] Suppose that $A \in \mathbb{C}^{n\times n}$ satisfies $A^2 = A$. Prove that A is diagonalizable.
(Hint: you may find some inspiration in a previous homework)

(5) Suppose that $A \in \mathbb{C}^{n\times n}$ has characteristic polynomial $P_A(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_k)^{m_k}$ where $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ are distinct; that is, A has eigenvalues $\lambda_1, \ldots, \lambda_k$ where λ_i has multiplicity m_i. Define $Q_A(t) = (t-\lambda_1)\cdots(t-\lambda_k)$.
 (a) [**+1**] Show that if A is diagonalizable, then $Q_A(A) = O_n$.
 (b) [**+1**] Find an example of some $A \in \mathbb{C}^{n\times n}$ for which $Q_A(A) \neq O_n$.

(6) [**+2**] Show that $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if there is an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{C}^n$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $A = \sum_{i=1}^n \lambda_i \vec{v}_i \vec{v}_i^*$.

(7) For a matrix $A \in \mathbb{C}^{n\times n}$, the *trace* of A is defined as $\operatorname{tr} A = \sum_{i=1}^n A_{ii}$; that is, the sum of the diagonal entries.
 (a) [**+1**] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$. Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
 (b) [**+1**] Let $A \in \mathbb{C}^{n\times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Show that $\operatorname{tr} A = \lambda_1 + \cdots + \lambda_n$.

(8) **Bonus**[+1] Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be *any* orthonormal basis for $\mathbb{C}^n$. Show that $\operatorname{tr} A = \sum_{k=1}^n \langle \vec{v}_i, A\vec{v}_i \rangle$ where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product.