Show your work and justify all answers.

(11 pts)

(1) [+1] Suppose that $A$ is a non-singular matrix with $A, A^{-1} \in \mathbb{Z}^{n \times n}$; that is both $A$ and $A^{-1}$ have only integer entries. What are the possible values for $\det A$?

(2) [+1] Show that if $n$ is odd, then $A - A^T$ is singular for all $A \in \mathbb{C}^{n \times n}$.

(3) [+1] Find the eigenvalues and eigenspaces of \[
\begin{pmatrix}
5 & -3 \\
6 & -4
\end{pmatrix}.
\]

(4) [+2] Suppose that $A \in \mathbb{C}^{n \times n}$ satisfies $A^2 = A$. Prove that $A$ is diagonalizable.

(Hint: you may find some inspiration in a previous homework)

(5) Suppose that $A \in \mathbb{C}^{n \times n}$ has characteristic polynomial $P_A(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_k)^{m_k}$ where $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ are distinct; that is, $A$ has eigenvalues $\lambda_1, \ldots, \lambda_k$ where $\lambda_i$ has multiplicity $m_i$. Define $Q_A(t) = (t - \lambda_1) \cdots (t - \lambda_k)$.

(a) [+1] Show that if $A$ is diagonalizable, then $Q_A(A) = O_n$.

(b) [+1] Find an example of some $A \in \mathbb{C}^{n \times n}$ for which $Q_A(A) \neq O_n$.

(6) [+2] Show that $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if there is an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{C}^n$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $A = \sum_{i=1}^n \lambda_i \vec{v}_i \vec{v}_i^*$.

(7) For a matrix $A \in \mathbb{C}^{n \times n}$, the trace of $A$ is defined as $\text{tr} A = \sum_{i=1}^n A_{ii}$; that is, the sum of the diagonal entries.

(a) [+1] Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Show that $\text{tr}(AB) = \text{tr}(BA)$.

(b) [+1] Let $A \in \mathbb{C}^{n \times n}$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). Show that $\text{tr} A = \lambda_1 + \cdots + \lambda_n$.

(8) **Bonus** [+1] Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be any orthonormal basis for $\mathbb{C}^n$. Show that $\text{tr} A = \sum_{k=1}^n \langle \vec{v}_k, A \vec{v}_k \rangle$ where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product.