Justify all answers! I recommend doing these questions out of order and focus first on questions with which you are less comfortable.

Please let me know if I've made any mistakes in my solutions.

(1) This exercise will walk through the last necessary step in our proof of the existence of Jordan canonical form; namely that the generalized eigenspaces associated with different eigenvalues are linearly independent.

Fix $A \in \mathbb{C}^{n\times n}$.

(a) Suppose that $\lambda, \mu \in \mathbb{C}$, $\vec{v} \in E_\lambda(A)$ and $k \in \mathbb{Z}_+$; show that $(A - \mu I_n)^k \vec{v} = (\lambda - \mu)^k \vec{v}$.

Solution: We notice that $(A - \mu I_n)\vec{v} = A\vec{v} - \mu\vec{v} = (\lambda - \mu)\vec{v}$. Thus, $\vec{v}$ is an eigenvector for $(A - \mu I_n)$ with eigenvalue $(\lambda - \mu)$. This implies that $\vec{v}$ is an eigenvector for $(A - \mu I_n)^k$ with eigenvalue $(\lambda - \mu)^k$ as needed. □

(b) Show that if $\lambda \neq \mu$, then $E^k_\lambda(A) \cap E^\ell_\mu(A) = \{\vec{0}\}$ for any $k, \ell \in \mathbb{Z}_+$.
(Hint: use part (a) by considering $(A - \lambda I_n)^{r-1}\vec{v}$ where r is the order of $\vec{v}$ as a generalized eigenvector associated with λ.)

Solution: If $k = 0$ or $\ell = 0$, then this is trivial, so suppose that $k, \ell \geq 1$.
Let $\vec{v} \in E^k_\lambda(A) \cap E^\ell_\mu(A)$; we wish to show that $\vec{v} = \vec{0}$. If not, then since $\vec{v} \in E^k_\lambda(A)$, let $r \geq 1$ be the order of $\vec{v}$ as a generalized eigenvector associated with λ, so $\vec{v}' = (A - \lambda I_n)^{r-1}\vec{v}$ has $\vec{v}' \in E_\lambda(A)$ and $\vec{v}' \neq \vec{0}$. Additionally, since $\vec{v} \in E^\ell_\mu(A)$ and polynomials in A commute, we know that $\vec{v}' \in E^\ell_\mu(A)$ still.
Thus, applying part (a), we have $\vec{0} = (A - \mu I_n)^\ell \vec{v}' = (\lambda - \mu)^\ell \vec{v}'$, so $(\lambda - \mu)^\ell = 0$ since $\vec{v}' \neq \vec{0}$; a contradiction since $\lambda \neq \mu$. □

(c) Suppose that $\vec{v}_1, \dots, \vec{v}_m$ are nonzero generalized eigenvectors for A, each associated with a different eigenvalue. Show that $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent.
(Hint: induction on m using part (b).)

Solution: Base case: $m = 1$ is trivial.
Induction hypothesis: For some $M > 1$, if $\vec{v}_1, \dots, \vec{v}_{M-1}$ are nonzero generalized eigenvectors, each associated with a different eigenvalue, then $\{\vec{v}_1, \dots, \vec{v}_{M-1}\}$ is linearly independent.
Induction step: Let $\vec{v}_1, \dots, \vec{v}_M$ be nonzero generalized eigenvectors where $\vec{v}_i$ is associated with λ_i, where $\lambda_1, \dots, \lambda_M$ are distinct. Suppose that $\vec{v}_i$ has order k_i as a generalized eigenvector associated with λ_i.
Now, consider a linear combination $c_1\vec{v}_1 + \cdots + c_M\vec{v}_M = \vec{0}$ and multiply by $(A - \lambda_M I_M)^{k_M}$, so find

$$c_1(A - \lambda_M I_M)^{k_M}\vec{v}_1 + c_2(A - \lambda_M I_M)^{k_M}\vec{v}_2 + \cdots + c_{M-1}(A - \lambda_M I_M)^{k_M}\vec{v}_{M-1} = \vec{0}.$$

Now, since $\lambda_i \neq \lambda_M$ for all $i \in [M-1]$, we know that $(A - \lambda_M I_M)^{k_M}\vec{v}_i \neq \vec{0}$ for all $i \in [M-1]$ by part (b) and also that $(A - \lambda_M I_M)^{k_M}\vec{v}_i$ is still a generalized eigenvector associated with λ_i since polynomials in A commute.
Thus, by the induction hypothesis, we see that $c_1 = \cdots = c_{M-1} = 0$, so $\vec{0} = \vec{0} + \cdots + \vec{0} + c_M\vec{v}_M$, so $c_M = 0$ as well since $\vec{v}_M \neq \vec{0}$. □

(2) For the following matrices A, find P, J where J is a Jordan form and $A = PJP^{-1}$.

(a) $\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$.

Solution: Observe that $P_A(t) = (t-2)^2$. Now,

$$E_2(A) = \operatorname{Nul} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \operatorname{span}\{\vec{e}_1\},$$

and

$$E_2^2(A) = \operatorname{Nul} O_2 = \mathbb{C}^2.$$

We can thus pick $\vec{v}_2 = \vec{e}_2$, and $\vec{v}_1 = (A - 2I_2)\vec{v}_2 = -\vec{e}_1$, so with $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ we have

$$A = P \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} P^{-1}.$$

□

(b) $\begin{bmatrix} 5 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Solution: Observe that $P_A(t) = (t-4)^3$. Now,

$$E_4(A) = \operatorname{Nul} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\left\{ \vec{e}_3, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

and

$$E_4^2(A) = \operatorname{Nul} O_3 = \mathbb{C}^3.$$

Now, notice that $\vec{v}_2 = \vec{e}_1$ is a basis for $E_4^2(A)$ relative to $E_4^1(A)$, and $\vec{v}_1 = (A - 4I_3) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.
Finally, with $\vec{v}_3 = \vec{e}_3$, $\{\vec{v}_2, \vec{v}_3\}$ forms a basis for $E_4(A)$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{C}^3$, so with $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$, we have

$$A = P \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} P^{-1}.$$

□

(3) Fix $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$. Suppose that $\ell \in \mathbb{Z}_+$ is such that $E_\lambda^{\ell-1}(A) \subsetneq E_\lambda^{\ell}(A) = E_\lambda^{\ell+1}(A)$, i.e. the chain of generalized λ-eigenspaces stabilizes at order ℓ. Prove that if $k \geq \ell$, then

$$\operatorname{Nul}\big((A - \lambda I_n)^k\big) + \operatorname{Col}\big((A - \lambda I_n)^k\big) = \mathbb{C}^n.$$

(Hint: Show that the intersection of these spaces is $\{\vec{0}\}$ and apply rank–nullity.)

Solution: By the rank–nulity theorem, we know that the dimensions of these spaces add to n, so it suffices to show that $\operatorname{Nul}\big((A - \lambda I_n)^k\big) \cap \operatorname{Col}\big((A - \lambda I_n)^k\big) = \{\vec{0}\}$.

Indeed, suppose that $\vec{v} \in \operatorname{Nul}\big((A - \lambda I_n)^k\big) \cap \operatorname{Col}\big((A - \lambda I_n)^k\big)$, so $(A - \lambda I_n)^k \vec{v} = \vec{0}$ and there is some $\vec{u} \in \mathbb{C}^n$ such that $(A - \lambda I_n)^k \vec{u} = \vec{v}$. Thus, $(A - \lambda I_n)^{2k} \vec{u} = (A - \lambda I_n)^k \vec{v} = \vec{0}$, so $\vec{u} \in E_\lambda^{2k}(A) = E_\lambda^k(A)$, since the chain stabilizes at order $\ell \leq k$. But, then, since $\vec{u} \in E_\lambda^k(A)$, we in fact have $\vec{v} = (A - \lambda I_n)^k \vec{u} = \vec{0}$, as needed. □

(4) Show that if $A \in \mathbb{C}^{n\times n}$ is diagonalizable, then $\operatorname{Nul} A \cap \operatorname{Col} A = \{\vec{0}\}$.

Solution: Since A is diagonalizable, we know that $E_0^1(A) = E_0^2(A)$; i.e. this chain stabilizes at order either 0 or 1 (depending on whether or not 0 is an eigenvalue for A). Therefore, this follows from work done above.

To reiterate the argument here in a simpler case: Suppose that $\vec{v} \in \operatorname{Nul} A \cap \operatorname{Col} A$, then $A\vec{v} = \vec{0}$ and there is some $\vec{u} \in \mathbb{C}^n$ with $A\vec{u} = \vec{v}$. Thus, $A^2\vec{u} = A\vec{v} = \vec{0}$, so $\vec{u} \in E_0^2(A) = E_0^1(A)$, so $\vec{v} = A\vec{u} = \vec{0}$. □