Justify all answers! I recommend doing these questions out of order and focus first on questions with which you are less comfortable.

Please let me know if I've made any mistakes in my solutions.

(1) Suppose that $A \in \mathbb{C}^{n\times n}$ is non-singular.

(a) Show that there is a polynomial p with $\deg p < n$ and $A^{-1} = p(A)$.

Solution: Let $P_A(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n$ be the characteristic polynomial for A. Since A is non-singular, we know that 0 is not an eigenvalue of A, so $a_0 \neq 0$, so write $P_A(t) = a_0 + tp(t)$ and notice that $\deg p < n$. By the Cayley–Hamilton theorem, we then have $O_n = P_A(A) = a_0 I_n + Ap(A)$, so since $a_0 \neq 0$, we have $A^{-1} = -\frac{1}{a_0}p(A)$. □

(b) Use part (a) to verify that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Solution: We first compute $P_A(t) = t^2 - (a+d)t + (ad-bc)$. By applying part (a), we thus have

$$A^{-1} = -\frac{1}{ad-bc}\big(A - (a+d)I_2\big) = -\frac{1}{ad-bc}\begin{bmatrix} a-(a+d) & 0-b \\ 0-c & d-(a+d) \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

□

(2) Suppose that $A \in \mathbb{C}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ which are all distinct. Must A be diagonalizable?

Solution: Yes. We know that there are some $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{C}^n$ with $A\vec{v}_i = \lambda_i \vec{v}_i$. Then by HW9, since $\lambda_1, \ldots, \lambda_n$ are distinct, we know that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent, and therefore a basis for $\mathbb{C}^n$. □

(3) Use Cayley–Hamilton to show that if $A \in \mathbb{C}^{2\times 2}$ has $\operatorname{tr} A = 0$, then $A^2 = cI_2$ for some $c \in \mathbb{C}$.

Solution: By direct computation we see that if $A \in \mathbb{C}^{2\times 2}$, then $P_A(t) = t^2 - (\operatorname{tr} A)t + \det A$, so $P_A(t) = t^2 + \det A$ in this case. Then by Caley–Hamilton, $O_2 = P_A(A) = A^2 + (\det A)I_2 \implies A^2 = -(\det A)I_2$. □

(4) Suppose that $A \in \mathbb{C}^{n\times n}$ satisfies $A^2 = I_n$. Prove that A is diagonalizable.

Solution: Notice that the fact that $A^2 = I_n$ means that all eigenvalues of A are either 1 or -1; hence we need to show that $E_1(A) + E_{-1}(A) = \mathbb{C}^n$. Let $\vec{x} \in \mathbb{C}^{n\times n}$; we need to write $\vec{x} = \vec{x}_1 + \vec{x}_{-1}$ where $\vec{x}_1 \in E_1(A)$ and $\vec{x}_{-1} \in E_{-1}(A)$. Set $\vec{x}_1 = \frac{1}{2}(\vec{x} + A\vec{x})$ and $\vec{x}_{-1} = \frac{1}{2}(\vec{x} - A\vec{x})$ so that $\vec{x}_1 + \vec{x}_{-1} = \vec{x}$.

Now, $A\vec{x}_1 = \frac{1}{2}(A\vec{x} + A^2\vec{x}) = \frac{1}{2}(A\vec{x} + \vec{x}) = \vec{x}_1$, so $\vec{x}_1 \in E_1(A)$.

On the other hand, $A\vec{x}_{-1} = \frac{1}{2}(A\vec{x} - A^2\vec{x}) = \frac{1}{2}(A\vec{x} - \vec{x}) = -\vec{x}_{-1}$, so $\vec{x}_{-1} \in E_{-1}(A)$. □

(5) Suppose that $A \in \mathbb{R}^{n\times n}_{>0}$, i.e. A has strictly positive entries. Suppose that all row-sums of A are 1; that is $A\vec{1} = \vec{1}$. Show that $E_1(A) = \operatorname{span}\{\vec{1}\}$.

Solution: Suppose that $A\vec{x} = \vec{x}$ with $\vec{x} \neq 0$; we need to show that $\vec{x}$ is a scalar multiple of $\vec{1}$; that is $x_1 = x_2 = \cdots = x_n$. Suppose that $k \in [n]$ $|x_k| \geq |x_i|$ for all $i \in [n]$; since we can scale an eigenvector to get another and $\vec{x} \neq \vec{0}$, we may suppose that $x_k = 1$.

Now, $A\vec{x} - \vec{x} = \vec{0}$, so by looking at the kth row of this equation, we find that

$$0 = (A_{k1}x_1 + A_{k2}x_2 + \cdots + A_{kn}x_n) - x_k \implies 1 = \sum_{\ell} A_{k\ell}x_\ell.$$

Now, if $\vec{x} \neq \vec{1}$, then since $|x_i| \leq 1$ for all i, there must be some i for which $|x_i| < 1$. Thus, since $A_{k\ell} > 0$ for all ℓ, we would then have

$$1 = |1| = \left|\sum_{\ell} A_{k\ell}x_\ell\right| \leq \sum_{\ell} A_{k\ell}|x_\ell| < \sum_{\ell} A_{k\ell} = 1;$$

a contradiction. $\square$

(6) A matrix $A \in \mathbb{C}^{n\times n}$ is called *nilpotent* if there is some integer k for which $A^k = O_n$. Show that if A is nilpotent, then in fact $A^n = O_n$.

Solution: Since $A^k = O_n$ for some k, we know that all eigenvalues of A are 0; in other words $P_A(t) = t^n$. Thus, by Caley–Hamilton, $O_n = P_A(A) = A^n$. $\square$

(7) Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix (recall that this is just the name for real, unitary matrices) with $\det Q = 1$. Show that if n is odd, then there is some non-zero $\vec{x} \in \mathbb{R}^n$ with $Q\vec{x} = \vec{x}$.
(Hint: what do you know about the roots of a polynomial with real coefficients?)

Solution: We simply need to show that 1 is an eigenvalue for Q.

$Q \in \mathbb{R}^{n\times n}$, we know that $P_Q(t)$ is a polynomial with real coefficients. The roots of such a polynomial come in conjugate pairs, so the eigenvalues are $\lambda_1, \overline{\lambda_1}, \ldots, \lambda_k, \overline{\lambda_k}, \lambda_{2k+1}, \ldots, \lambda_n$, where $\lambda_{2k+1}, \ldots, \lambda_n \in \mathbb{R}$.

Now, $\lambda_i\overline{\lambda_i} = |\lambda_i|^2 = 1$ since Q is orthogonal, so $1 = \det Q = (\lambda_1\overline{\lambda_1})\cdots(\lambda_k\overline{\lambda_k})\lambda_{2k+1}\cdots\lambda_n = \lambda_{2k+1}\cdots\lambda_n$.

Now, $\lambda_{2k+1}, \ldots, \lambda_n \in \{\pm 1\}$ since they are real, so since $\lambda_{2k+1}\cdots\lambda_n = 1$, we know that there must be an even number of -1's. Hence, since n is odd, we have $n - (2k+1) + 1 = n - 2k + 2$ of these λ's, which is an odd number; hence Q does indeed have 1 as an eigenvalue. $\square$

(8) Let $A, B \in \mathbb{C}^{n\times n}$.

(a) Show that if A (or B) is non-singular, then $P_{AB}(t) = P_{BA}(t)$.

Solution: We claim that AB and BA are similar in this case, from which the claim follows. Indeed, since A is non-singular, $AB = A(BA)A^{-1}$. $\square$

(b) **Challenge:** Show that the same is true if A, B are singular. (Hint: "perturb" A by a small multiple of the identity)

Solution: In DSW4, there was an example of singular A, B where AB was not similar to BA, so we will need a different argument here, namely one that uses continuity.

Firstly, for $\gamma \in \mathbb{C}$, consider $A_\gamma = A - \gamma I_n$. A_γ is simply a polynomial in A, so we know that the eigenvalues of A_γ are $\lambda_i - \gamma$ where λ_i is an eigenvalue for A. Since there are only finitely many eigenvalues, there is some $\epsilon > 0$ such that for all $0 < |\gamma| < \epsilon$, all eigenvalues of A_γ are non-zero, i.e. A_γ is non-singular.

Thus, by part (a), we know that $P_{A_\gamma B}(t) = P_{BA_\gamma}(t)$ for all $0 < |\gamma| < \epsilon$. Now, $P_{A_\gamma B}(t) = \det(tI_n - (A - \gamma I_n)B) = \det(tI_n - AB + \gamma B)$ and $P_{BA_\gamma}(t) = \det(tI_n - BA + \gamma B)$, so

$$\lim_{\gamma\to 0} \det(tI_n - AB + \gamma B) = \lim_{\gamma\to 0} \det(tI_n - BA + \gamma B).$$

We now claim that $\lim_{\gamma\to 0}\det(X+\gamma Y)=\det X$ for any $X, Y\in\mathbb{C}^{n\times n}$. To show this, we expand $\det(X+\gamma Y)=\sum_{\sigma\in S_n}\operatorname{sgn}\sigma\prod_{i=1}^n(X_{i,\sigma(i)}+\gamma Y_{i,\sigma(i)})$. But now, $X_{i,\sigma(i)}, Y_{i,\sigma(i)}$ are just numbers, so $\lim_{\gamma\to 0}(X_{i,\sigma(i)}+\gamma Y_{i,\sigma(i)}))=X_{i,\sigma(i)}$, and hence since there are only finitely many permutations in S_n, $\lim_{\gamma\to 0}\det(X+\gamma Y)=\det X$ as needed.

We conclude that

$$P_{AB}(t)=\det(tI_n-AB)=\lim_{\gamma\to 0}\det(tI_n-AB+\gamma B)=\lim_{\gamma\to 0}\det(tI_n-BA+\gamma B)=\det(tI_n-BA)=P_{BA}(t).$$

□

(9) Suppose that $A\succ 0$ and $B\succeq 0$. Show that if $AB=BA$, then $AB\succeq 0$.

Solution: Since A, B are Hermitian and $AB=BA$, we have $(AB)^*=B^*A^*=BA=AB$, so AB is Hermitian. We showed in class that the eigenvalues of AB are non-negative, so the two of these together means that $AB\succeq 0$. □