Justify all answers! I recommend doing these questions out of order and focus first on questions with which you are less comfortable.

(1) Find a matrix representation for the following linear transformations. Write your matrix with respect to the standard basis.

(a) Reflection of $\mathbb{R}^2$ across the line $y = -2x$.

Solution: Let R denote the transformation. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so $\mathcal{B}$ is a basis for $\mathbb{R}^2$. Notice that $R(\vec{b}_1) = \vec{b}_1$ and $R(\vec{b}_2) = -\vec{b}_2$, so $[M_R]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Therefore,

$$M_R = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

□

(b) Rotation of $\mathbb{R}^2$ by $\pi/2$ radians anti-clockwise and then a projection onto the line $3y = 4x$.

Solution: Let R be the rotation and P be the projection. Observe that $R(\vec{e}_1) = \vec{e}_2$ and $R(\vec{e}_2) = -\vec{e}_1$, so $M_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Now for P. Let $\vec{b}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and notice that $\mathcal{B} = \{\frac{1}{5}\vec{b}_1, \frac{1}{5}\vec{b}_2\}$ is an orthonormal basis for $\mathbb{R}^2$. Additionally, $P(\vec{b}_1) = \vec{b}_1$ and $P(\vec{b}_2) = \vec{0}$, so $[M_P]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, if $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$, we have

$$M_P = B[M_P]_{\mathcal{B}}B^{-1} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{25} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

Therefore, the full transformation is represented by the matrix

$$M_{P \circ R} = M_P M_R = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 12 & -9 \\ 16 & -12 \end{bmatrix}.$$

□

(2) Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and suppose that $A\vec{x} = \lambda\vec{x}$ where $\lambda \in \mathbb{C}$ and $\vec{x} \neq \vec{0}$, i.e. $\vec{x}$ is an eigenvector with eigenvalue λ.

(a) Show that $\lambda \in \mathbb{R}$. (Hint: consider $\langle \vec{x}, A\vec{x} \rangle$)

Solution: We first notice that $\langle \vec{x}, A\vec{x} \rangle = \langle \vec{x}, \lambda\vec{x} \rangle = \lambda\langle \vec{x}, \vec{x} \rangle$.
On the other hand, since A is Hermitian, $\langle \vec{x}, A\vec{x} \rangle = \langle A^*\vec{x}, \vec{x} \rangle = \langle A\vec{x}, \vec{x} \rangle = \langle \lambda\vec{x}, \vec{x} \rangle = \overline{\lambda}\langle \vec{x}, \vec{x} \rangle$.
Therefore, $\lambda\langle \vec{x}, \vec{x} \rangle = \overline{\lambda}\langle \vec{x}, \vec{x} \rangle$. Since $\vec{x} \neq \vec{0}$, we know that $\langle \vec{x}, \vec{x} \rangle \neq 0$, so $\lambda = \overline{\lambda}$, meaning that $\lambda \in \mathbb{R}$.
□

(b) Suppose also that $A\vec{y} = \mu\vec{y}$ where $\vec{y} \neq \vec{0}$ and $\mu \neq \lambda$. Show that $\langle \vec{x}, \vec{y} \rangle = 0$.

Solution: We first notice that $\langle \vec{x}, A\vec{y} \rangle = \langle \vec{x}, \mu\vec{y} \rangle = \mu\langle \vec{x}, \vec{y} \rangle$. On the other hand, since A is Hermitian, we have $\langle \vec{x}, A\vec{y} \rangle = \langle A^*\vec{x}, \vec{y} \rangle = \langle A\vec{x}, \vec{y} \rangle = \langle \lambda\vec{x}, \vec{y} \rangle = \overline{\lambda}\langle \vec{x}, \vec{y} \rangle$.

Therefore, $\overline{\lambda}\langle\vec{x},\vec{y}\rangle = \mu\langle\vec{x},\vec{y}\rangle$, so if $\langle\vec{x},\vec{y}\rangle \neq 0$, we must have $\overline{\lambda} = \mu$. However, by part (a), we know that $\lambda, \mu \in \mathbb{R}$, so this implies that $\lambda = \mu$; contradicting our assumption. ☐

(3) Prove the following statements related to similarity of matrices.

(a) If A is similar to B, then A^* is similar to B^* and A^T is similar to B^T.

Solution: We have $A = PBP^{-1}$, so $A^* = (P^{-1})^* B^* P^* = (P^*)^{-1} B^* P^*$, so A^* is similar to B^* since $P^* = ((P^*)^{-1})^{-1}$. Similarly[1], $A^T = (P^T)^{-1} B^T P^T$, so A^T is similar to B^T. ☐

(b) If A is similar to B and B is similar to C, then A is similar to C.

Solution: We have $A = PBP^{-1}$ and $B = QCQ^{-1}$, so $A = PQCQ^{-1}P^{-1} = (PQ)C(PQ)^{-1}$, so A is similar to C. ☐

(c) If A is similar to B, then $\operatorname{rank} A = \operatorname{rank} B$.

Solution: We have $A = PBP^{-1}$ where P is non-singular. We know that if C is non-singular, then $\operatorname{rank}(CB) = \operatorname{rank}(BC) = \operatorname{rank} B$, so $\operatorname{rank} A = \operatorname{rank}(PBP^{-1}) = \operatorname{rank}(PB) = \operatorname{rank} B$. ☐

(d) Find matrices A, B such that AB is *not* similar to BA.

Solution: Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Compute $AB = O_2$ and $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Since $\operatorname{rank}(AB) \neq \operatorname{rank}(BA)$, we know that AB cannot be similar to BA. ☐

(e) If A is non-singular, then AB is similar to BA.

Solution: Since A is non-singular, A^{-1} exists, so $AB = A(BA)A^{-1}$, i.e. AB is similar to BA. ☐

(f) If A is diagonalizable, then A is similar to A^T.

Solution: A is diagonalizable, so $A = PDP^{-1}$ where D is diagonal. Isolating D, we find that $D = P^{-1}AP$ and since D is diagonal, $D = D^T = P^T A^T (P^T)^{-1}$. Therefore, $A^T = (P^T)^{-1} D P^T = (P^T)^{-1} P^{-1} A P P^T = (PP^T)^{-1} A (PP^T)$, so A^T is similar to A. ☐

(g) If A is unitarily similar to B, then A^*A is unitarily similar to B^*B.

Solution: We have $A = UBU^*$ for some unitary matrix U, so $A^*A = (UBU^*)^*(UBU^*) = UB^*U^*UBU^* = UB^*BU^*$. ☐

(4) Consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product. We have proved multiple times that if $S \leq \mathbb{C}^n$ and $\{\vec{s}_1, \dots, \vec{s}_k\}$ is an orthonormal basis for S, then $\operatorname{proj}_S = AA^*$ where $A = \begin{bmatrix} \vec{s}_1 & \cdots & \vec{s}_k \end{bmatrix}$. Give yet another proof of this fact which uses a change-of-basis.

Solution: Extend $\{\vec{s}_1, \dots, \vec{s}_k\}$ to an orthonormal basis $\mathcal{B} = \{\vec{s}_1, \dots, \vec{s}_k, \vec{s}_{k+1}, \dots, \vec{s}_n\}$ and notice that

$$[\operatorname{proj}_S]_{\mathcal{B}} = \begin{bmatrix} I_k & O_{k \times n-k} \\ O_{n-k \times k} & O_{n-k} \end{bmatrix}$$

Now, with $B = \begin{bmatrix} \vec{s}_1 & \cdots & \vec{s}_n \end{bmatrix}$, we note that $B^{-1} = B^*$ since $\{\vec{s}_1, \dots, \vec{s}_n\}$ is an orthonormal basis. Thus,

$$\operatorname{proj}_S = B[\operatorname{proj}_S]_{\mathcal{B}} B^* = AA^*,$$

since the entries corresponding to $\vec{s}_{k+1}, \dots, \vec{s}_n$ in B are cancelled out by the 0's in $[\operatorname{proj}_S]_{\mathcal{B}}$. ☐

[1] Pun very much intended.