Justify all answers! I recommend doing these questions out of order and focus first on questions with which you are less comfortable.

(1) Recall that $A \in \mathbb{C}^{n\times n}$ is called *Hermitian* if $A^* = A$. Show that if A is Hermitian, then $\langle \vec{x}, A\vec{x}\rangle \in \mathbb{R}$ for all $\vec{x} \in \mathbb{C}^n$, where $\langle \cdot, \cdot\rangle$ is the standard Hermitian inner product.

Solution: By problem (d) on HW6, we know that $\langle \vec{x}, A\vec{x}\rangle = \langle A\vec{x}, \vec{x}\rangle$ for all $\vec{x} \in \mathbb{C}^n$. On the other hand, $\langle A\vec{x}, \vec{x}\rangle = \overline{\langle \vec{x}, A\vec{x}\rangle}$, so $\langle \vec{x}, A\vec{x}\rangle = \overline{\langle \vec{x}, A\vec{x}\rangle}$, which means that $\langle \vec{x}, A\vec{x}\rangle \in \mathbb{R}$. □

(2) Fix $A \in \mathbb{R}^{m\times n}$ and let $\langle \cdot, \cdot\rangle$ be the standard Euclidean inner product. Determine $(\operatorname{Nul} A)^\perp$.

Solution: We claim that $(\operatorname{Nul} A)^\perp = \operatorname{Row} A = \operatorname{Col}(A^T)$.

We know that $\operatorname{Nul} A = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$. In other words, if $\vec{a}_i$ is the ith row of A, then we know that $\langle \vec{a}_i, \vec{x}\rangle = 0$ whenever $\vec{x} \in \operatorname{Nul} A$, so $(\operatorname{Nul} A)^\perp \supseteq \operatorname{Row} A$. Now, since $A \in \mathbb{R}^{m\times n}$, we know that $\dim \operatorname{Nul} A + \dim(\operatorname{Nul} A)^\perp = n = \dim \operatorname{Nul} A + \dim \operatorname{Row} A$, so $\dim \operatorname{Row} A = \dim(\operatorname{Nul} A)^\perp$, so since all of these spaces are finite dimensional and $\operatorname{Row} A \subseteq (\operatorname{Nul} A)^\perp$, we must have $(\operatorname{Nul} A)^\perp = \operatorname{Row} A$. □

(3) Fix $A \in \mathbb{C}^{m\times n}$ and let $\langle \cdot, \cdot\rangle$ be the standard Hermitian inner product. Determine $(\operatorname{Nul} A)^\perp$.

Solution: We claim that $(\operatorname{Nul} A)^\perp = \operatorname{Col}(A^*) = \operatorname{Row}(\overline{A})$.

Since $\operatorname{Nul} A = \{\vec{x} \in \mathbb{C}^n : A\vec{x} = \vec{0}\}$, we observe that if $\vec{a}_i$ is a column of A^*, then $\vec{a}_i^*$ is a row of A, so by the same reasoning as problem (2), we see that $\langle \vec{a}_i, \vec{x}\rangle = \vec{a}_i^*\vec{x} = 0$, so $\operatorname{Col}(A^*) \subseteq (\operatorname{Nul} A)^\perp$. The full conclusion again follows via the rank–nullity theorem as in the previous problem. □

(4) Let V be a finite-dimensional inner product space and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis.

(a) Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Show that $\sum_{\ell=1}^n |\langle v_\ell, x\rangle|^2 = \|x\|^2$ for any $x \in V$.

Solution: We can write $x = \sum_{\ell=1}^n \langle v_\ell, x\rangle v_\ell$, so

$$\begin{aligned}\|x\|^2 = \left\langle \sum_{\ell=1}^n \langle v_\ell, x\rangle v_\ell,\ \sum_{\ell=1}^n \langle v_\ell, x\rangle v_\ell \right\rangle &= \sum_{t=1}^n \langle v_t, x\rangle \left\langle \sum_{\ell=1}^n \langle v_\ell, x\rangle v_\ell,\ v_t \right\rangle \\ &= \sum_{t,\ell} \langle v_t, x\rangle \overline{\langle v_\ell, x\rangle} \langle v_\ell, v_t\rangle = \sum_{\ell=1}^n |\langle v_\ell, x\rangle|^2.\end{aligned}$$

□

(b) Let $\{v_1, \ldots, v_k\}$ be an orthonormal set (not necessarily a basis). Show that $\sum_{\ell=1}^k |\langle v_\ell, x\rangle|^2 \le \|x\|^2$ for any $x \in V$. This is known as Bessel's inequality.

Solution: Suppose that $\dim V = n$, so through the Gram–Schmidt algorithm, we can extend $\{v_1, \ldots, v_k\}$ to an orthonormal basis $\{v_1, \ldots, v_n\}$. By part (a) and the fact that $|c|^2 \ge 0$ for any $c \in \mathbb{C}$, we bound

$$\|x\|^2 = \sum_{\ell=1}^n |\langle v_\ell, x\rangle|^2 \ge \sum_{\ell=1}^k |\langle v_\ell, x\rangle|^2.$$

□

(5) A collection of vectors $\vec{x}_1, \ldots, \vec{x}_m \in \mathbb{R}^n$ is called *equidistant* if there is some $d > 0$ for which $\|\vec{x}_i - \vec{x}_j\| = d$ for all $i \ne j$, where $\|\cdot\|$ is the standard Euclidean norm.

(a) Find a collection of $n+1$ equidistant vectors in $\mathbb{R}^n$.[1]
(Hint: consider the standard basis and some multiple of the all-ones vector)

Solution: Observe that for any $i \neq j \in [n]$, we have $\|\vec{e}_i - \vec{e}_j\|^2 = 2$. Furthermore, for any $\gamma \in \mathbb{R}$ and $i \in [n]$, we have $\|\vec{e}_i - \gamma\vec{1}\|^2 = (n-1)\gamma^2 + (1-\gamma)^2$. We claim that there is a choice of γ for which $(n-1)\gamma^2 + (1-\gamma)^2 = 2$, which would mean that $\{\vec{e}_1, \ldots, \vec{e}_n, \gamma\vec{1}\}$ is equidistant.
We could explicitly solve for such a γ, but that's no fun! Instead, set $f(\gamma) = (n-1)\gamma^2 + (1-\gamma)^2$ and notice that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Now, $f(0) = 1 < 2$ and $\lim_{\gamma\to\infty} f(\gamma) = \infty$, so by the intermediate value theorem, there must be some $\gamma \in \mathbb{R}$ for which $f(\gamma) = 2$. □

(b) For vectors $\vec{x}_1, \ldots, \vec{x}_m \in \mathbb{C}^n$, the *Gram matrix* is the matrix $G \in \mathbb{C}^{m\times m}$ where $G_{ij} = \langle \vec{x}_i, \vec{x}_j\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard Hermitian inner product. Prove that if G is the Gram matrix of some collection of vectors $\vec{x}_1, \ldots, \vec{x}_m \in \mathbb{C}^n$, then $\operatorname{rank} G \le n$.

Solution: Set $A = \begin{bmatrix}\vec{x}_1 & \cdots & \vec{x}_m\end{bmatrix}$, then $G = A^*A$; hence $\operatorname{rank} G = \operatorname{rank} A$. Now, A has n rows, so $\operatorname{rank} G = \operatorname{rank} A \le n$. □

(c) Show that there can be no more than $n+1$ equidistant vectors in $\mathbb{R}^n$.
(Hint: Without loss of generality, suppose that one of the vectors is $\vec{0}$ and determine the inner products and norms of the remaining vectors. Then apply part (b).)

Solution: Suppose that $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_m \in \mathbb{R}^n$ are equidistant and $\vec{x}_0 = \vec{0}$. Also, without loss of generality, the common distance is 1. Therefore, for $i \in [m]$, we observe that $\|\vec{x}_i\|^2 = \|\vec{x}_i - \vec{x}_0\|^2 = 1$. Thus, for $i \neq j \in [m]$, we have

$$\begin{aligned} 1 &= \|\vec{x}_i - \vec{x}_j\|^2 = \langle \vec{x}_i - \vec{x}_j, \vec{x}_i - \vec{x}_j\rangle = \|\vec{x}_i\|^2 - 2\langle \vec{x}_i, \vec{x}_j\rangle + \|\vec{x}_j\|^2 \\ &= 2 - 2\langle \vec{x}_i, \vec{x}_j\rangle \\ \implies \langle \vec{x}_i, \vec{x}_j\rangle &= 1/2. \end{aligned}$$

Let G be the Gram matrix of $\vec{x}_1, \ldots, \vec{x}_m$, so by above we know that G has 1's on the diagonal and 1/2's on the off-diagonal.
By the work done in our proof of Fisher's theorem, such a G has full rank, so $\operatorname{rank} G = m$. But then by part (b), we know that $m = \operatorname{rank} G \le n$. Thus, $m \le n$, so there were at most $n+1$ vectors to begin with. □

(d) Can you give an upper bound on the size of an equidistant set in $\mathbb{C}^n$?

Solution: We can give an upper bound of $2n+1$. Following the same steps as in part (c), we again find that $\|\vec{x}_i\|^2 = 1$ for all $i \in [m]$, but we now find that for $i \neq j \in [m]$,

$$1 = \langle \vec{x}_i, \vec{x}_j\rangle + \langle \vec{x}_j, \vec{x}_i\rangle.$$

We thus see that if G is the Gram matrix of $\vec{x}_1, \ldots, \vec{x}_m$, then $G + G^T$ has 2's on the diagonal and 1's on the off-diagonal, and therefore $\operatorname{rank}(G + G^T) = m$. Now, $\operatorname{rank} G = \operatorname{rank}(G^T) \le n$, so by problem (1a) on HW5, we see that $m = \operatorname{rank}(G + G^T) \le 2n$, so in all, there were at most $2n+1$ vectors to begin with. □

[1] Such a set forms the vertices of what is called a regular simplex, which is the higher-dimensional analogue of an equilateral triangle and equilateral tetrahedron.

(e) [$+\infty$ **bonus points**] Instead of the Euclidean norm, consider the ℓ_1-norm from HW6. Notice that $\{\pm\vec{e}_1, \dots, \pm\vec{e}_n\}$ is equidistant under $\|\cdot\|_1$ with common distance 2, so we can have sets of $2n$ vectors in $\mathbb{R}^n$ which are ℓ_1-equidistant. Can there be any more than $2n$ such vectors?

Solution: No one knows the answer to this question for $n \geq 6$. Currently, the best known upper bound is $Cn\log n$ for some fixed constant C. □

(6) Consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product. Let $S \leq \mathbb{C}^n$ and let $\{\vec{s}_1, \dots, \vec{s}_k\}$ be any basis for S (not necessarily orthonormal). Set $A = \begin{bmatrix} \vec{s}_1 & \cdots & \vec{s}_k \end{bmatrix} \in \mathbb{C}^{n\times k}$.

(a) Show that $\operatorname{proj}_S \vec{v} = \vec{p}$ if

1. $A^*\vec{v} = A^*\vec{p}$ and
2. $A\vec{x} = \vec{p}$ for some $\vec{x} \in \mathbb{C}^k$.

(Recall that, in general, $\operatorname{proj}_S v = p$ if $p \in S$ and $v - p \in S^\perp$.)

Solution: Using the hint in the parentheses, $\operatorname{proj}_S \vec{v} = \vec{p}$ if $\vec{p} \in S$ and $\vec{v} - \vec{p} \in S^\perp$. Certainly $\vec{p} \in S$ if and only if $\vec{p} \in \operatorname{Col} A$, i.e. $A\vec{x} = \vec{p}$ has a solution, and $\vec{v} - \vec{p} \in S^\perp$ if and only if $A^*(\vec{v} - \vec{p}) = \vec{0}$, i.e. $A^*\vec{v} = A^*\vec{p}$. □

(b) Show that $\operatorname{proj}_S \vec{v} = A(A^*A)^{-1}A^*\vec{v}$. Be sure to justify why $(A^*A)^{-1}$ exists.

Solution: We first note that $A \in \mathbb{C}^{n\times k}$ and $\operatorname{rank} A = k$ since $\dim S = k$. Thus, $A^*A \in \mathbb{C}^{k\times k}$ and $\operatorname{rank}(A^*A) = \operatorname{rank} A = k$, so A^*A is indeed non-singular.
Now, by part (a), we know that $\operatorname{proj}_S \vec{v} = \vec{p}$ if $A^*\vec{v} = A^*\vec{p}$ and $A\vec{x} = \vec{p}$ for some $\vec{x} \in \mathbb{C}^k$.
Substitute the second equation into the first to find that $\operatorname{proj}_S \vec{v} = A\vec{x}$ whenever $A^*A\vec{x} = A^*\vec{v}$. Since A^*A is non-singular, we can solve for $\vec{x} = (A^*A)^{-1}A^*\vec{v}$, and thus $\operatorname{proj}_S \vec{v} = A\vec{x} = A(A^*A)^{-1}A^*\vec{v}$.
□

(7) This is an interesting difference between the Euclidean and Hermitian inner products.

(a) Find a non-zero matrix $A \in \mathbb{R}^{2\times 2}$ such that $\langle \vec{x}, A\vec{x}\rangle = 0$ for all $\vec{x} \in \mathbb{R}^2$. Here $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product.

Solution: Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Observe that $A\vec{x} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$, so $\langle \vec{x}, A\vec{x}\rangle = x_1x_2 - x_2x_1 = 0$. □

(b) **Challenge:** Show that the same is impossible for the Hermitian inner product. That is, show that if $A \in \mathbb{C}^{n\times n}$ has the property that $\langle \vec{x}, A\vec{x}\rangle = 0$ for all $\vec{x} \in \mathbb{C}^n$, then $A = O_n$.
(Hint: Show that this implies that $\langle \vec{x}, A\vec{y}\rangle = 0$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$ by considering a linear combination of $\vec{x}$ and $\vec{y}$. Then apply problem (3b) from HW6.)

Solution: Let $\lambda \in \mathbb{C}$ be non-zero and $\vec{x}, \vec{y} \in \mathbb{C}^n$. We see that

$$\begin{aligned} 0 &= \langle \lambda\vec{x} + \vec{y}, A(\lambda\vec{x} + \vec{y})\rangle = \bar{\lambda}\lambda\langle \vec{x}, A\vec{x}\rangle + \bar{\lambda}\langle \vec{x}, A\vec{y}\rangle + \lambda\langle \vec{y}, A\vec{x}\rangle + \langle \vec{y}, A\vec{y}\rangle \\ &= \bar{\lambda}\langle \vec{x}, A\vec{y}\rangle + \lambda\langle \vec{y}, A\vec{x}\rangle \\ \implies \langle \vec{x}, A\vec{y}\rangle &= (\lambda/\bar{\lambda})\langle \vec{y}, A\vec{x}\rangle, \end{aligned}$$

since $\lambda \neq 0$. Suppose for the sake of contradiction that $\langle \vec{x}, A\vec{y}\rangle \neq 0$, so we must also have $\langle \vec{y}, A\vec{x}\rangle \neq 0$. Hence, for *any* non-zero $\lambda \in \mathbb{C}$, we must have

$$\frac{\langle \vec{x}, A\vec{y}\rangle}{\langle \vec{y}, A\vec{x}\rangle} = \frac{\lambda}{\bar{\lambda}},$$

implying that $\lambda/\bar{\lambda}$ is constant for all non-zero $\lambda \in \mathbb{C}$. This is, of course, absurd since $1/\bar{1} = 1$ while $i/\bar{i} = -1$.

Thus, $\langle \vec{x}, A\vec{y} \rangle = 0$ for all $\vec{x}, \vec{y} \in \mathbb{C}^n$, so we find that $A = O_n$ by problem (3b) on HW6. $\square$