Justify all answers! I recommend doing these questions out of order and focus first on questions with which you are less comfortable.

Throughout, V is assumed to be a vector space over $\mathbb{R}$.

Please let me know if I've made any mistakes in my solutions.

(1) Let $X \subseteq V$. Show that if $S \leq V$ contains X, then $S \supseteq \operatorname{span} X$. (This justifies the remark that $\operatorname{span} X$ is the "smallest" subspace which contains X)

Solution: If $S \leq V$ and $x_1, \ldots, x_n \in S$, then $c_1x_1 + \cdots + c_nx_n \in S$ for any $c_1, \ldots, c_n \in \mathbb{R}$.

Thus, if $X \subseteq S$, then S contains every finite linear combination of elements of X, so $\operatorname{span} X \subseteq S$ as well. $\square$

(2) This exercise will give us another way to define $\operatorname{span} X$.

(a) Let $\mathcal{S}$ be a collection of subspaces of V; that is, $\mathcal{S}$ is a set whose elements are subspaces of V. Show that $\bigcap_{S \in \mathcal{S}} S \leq V$ where $\bigcap_{S \in \mathcal{S}} S = \{v \in V : v \in S \text{ for every } S \in \mathcal{S}\}$. (This *cannot* be proved by induction since $\mathcal{S}$ may have infinitely many elements)

Solution: Set $\widehat{S} = \bigcap_{S \in \mathcal{S}} S$. Since $S \leq V$ for all $S \in \mathcal{S}$, we know that $0 \in S$ for all $S \in \mathcal{S}$. Therefore $0 \in \widehat{S}$, so $\widehat{S} \neq \varnothing$.

Now, take $x_1, x_2 \in \widehat{S}$ and $c_1, c_2 \in \mathbb{R}$; we need to show that $c_1x_1 + c_2x_2 \in \widehat{S}$. Since $x_1, x_2 \in \widehat{S}$, we know that $x_1, x_2 \in S$ for every $S \in \mathcal{S}$ and since $S \leq V$, we know that $c_1x_1 + c_2x_2 \in S$ as well. Therefore, $c_1x_1 + c_2x_2 \in S$ for every $S \in \mathcal{S}$, so $c_1x_1 + c_2x_2 \in \widehat{S}$. $\square$

(b) Fix $X \subseteq V$ and let $\mathcal{X}$ be the collection of all subspaces of V which contain X; that is, $\mathcal{X} = \{S \leq V : S \supseteq X\}$. Set $\widehat{X} = \bigcap_{S \in \mathcal{X}} S$.

Show that $\widehat{X} \supseteq X$ and that if $T \in \mathcal{X}$, then $T \supseteq \widehat{X}$.

Solution: Fix $x \in X$. By definition, $x \in S$ for every $S \in \mathcal{X}$, so $x \in \widehat{X}$ as well. Thus, $X \subseteq \widehat{X}$.

Fix $x \in \widehat{X}$; we need to show that $x \in T$. Since $x \in \widehat{X}$, we know that $x \in S$ for all $S \in \mathcal{X}$ by definition. Since $T \in \mathcal{X}$, this means that $x \in T$; thus $\widehat{X} \subseteq T$. $\square$

(c) Show that $\operatorname{span} X = \widehat{X}$.

Solution: We know that $\operatorname{span} X \in \mathcal{X}$, so by part (b), we must have $\operatorname{span} X \supseteq \widehat{X}$.

On the other hand, since $\widehat{X} \leq V$ and $\widehat{X} \supseteq X$, by problem (1), we know that $\widehat{X} \supseteq \operatorname{span} X$.

Therefore $\operatorname{span} X = \widehat{X}$. $\square$

(3) Show that if $X, Y \subseteq V$, then $\operatorname{span}(X \cup Y) = \operatorname{span} X + \operatorname{span} Y$.

Solution: We show first that $\operatorname{span} X + \operatorname{span} Y \subseteq \operatorname{span}(X \cup Y)$. Fix $z \in \operatorname{span} X + \operatorname{span} Y$, so we know we can write $z = x + y$ for some $x \in \operatorname{span} X$ and $y \in \operatorname{span} Y$. By the definition of span, we can write $x = c_1x_1 + \cdots + c_nx_n$ and $y = d_1y_1 + \cdots + d_my_m$ for some $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_m \in Y$, and $c_1, \ldots, c_n, d_1, \ldots, d_m \in \mathbb{R}$. Therefore $z = c_1x_1 + \cdots + c_nx_n + d_1y_1 + \cdots + d_my_m$, so $z \in \operatorname{span}(X \cup Y)$ since $x_1, \ldots, x_n, y_1, \ldots, y_m \in X \cup Y$.

For the other direction, fix $z \in \operatorname{span}(X \cup Y)$, so we can write $z = c_1z_1 + \cdots + c_nz_n$ for some $z_1, \ldots, z_n \in X \cup Y$ and $c_1, \ldots, c_n \in \mathbb{R}$. Now, relabel these z_i's and c_i's so that $z_1, \ldots, z_k \in X$ and $z_{k+1}, \ldots, z_n \in Y$ for some $0 \leq k \leq n$. Set $x = c_1z_1 + \cdots + c_kz_k$ and $y = c_{k+1}z_{k+1} + \cdots + c_nz_n$ where

$x = 0$ if $k = 0$ and $y = 0$ if $k = n$. Notice that $x \in \operatorname{span} X$ and $y \in \operatorname{span} Y$, so $z = x + y \in \operatorname{span} X + \operatorname{span} Y$. □

(4) Show that if $S_1, \dots, S_n \leq V$ are finite-dimensional subspaces, then $\dim(S_1 + \cdots + S_n) \leq \dim S_1 + \cdots + \dim S_n$.

Solution: We first prove by induction that if $X_1, \dots, X_n \subseteq V$, then $\operatorname{span}(X_1 \cup \cdots \cup X_n) = \operatorname{span} X_1 + \cdots + \operatorname{span} X_n$.

Base Cases: $n = 1$ is trivial.

$n = 2$ is proved in problem (3).

Induction Hypothesis: For some $N > 2$, for any $X_1, \dots, X_{N-1} \subseteq V$, we have $\operatorname{span}(X_1 \cup \cdots \cup X_{N-1}) = \operatorname{span} X_1 + \cdots + \operatorname{span} X_{N-1}$.

Induction Step: Let $X_1, \dots, X_N \subseteq V$ and set $Y = X_1 \cup \cdots \cup X_{N-1}$. By the $n = 2$ case, we know that $\operatorname{span}(Y \cup X_N) = \operatorname{span} Y + \operatorname{span} X_N$. Furthermore, by the induction hypothesis, $\operatorname{span} Y = \operatorname{span} X_1 + \cdots + \operatorname{span} X_{N-1}$. Therefore, $\operatorname{span}(X_1 \cup \cdots \cup X_N) = \operatorname{span} Y + \operatorname{span} X_N = \operatorname{span} X_1 + \cdots + \operatorname{span} X_{N-1} + \operatorname{span} X_N$ as needed.

Now, let $\mathcal{B}_i$ be a basis for S_i. We just showed that $S_1 + \cdots + S_n = \operatorname{span}(\mathcal{B}_1 \cup \dots \mathcal{B}_n)$, so we have

$$\dim(S_1 + \cdots + S_n) \leq |\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n| \leq |\mathcal{B}_1| + \cdots + |\mathcal{B}_n| = \dim S_1 + \cdots + \dim S_n.$$

□

(5) Let $S_1, S_2 \leq V$.

(a) Let $\mathcal{B}_i$ be a basis for S_i. Prove that if $S_1 \cap S_2 = \{0\}$, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $S_1 + S_2$.

Solution: We need to show that $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent and $\operatorname{span}(\mathcal{B}_1 \cup \mathcal{B}_2) = S_1 + S_2$. The latter follows directly from the previous problem, so we need only show linear independence.
Fix $x_1, \dots, x_n, y_1, \dots, y_m \in \mathcal{B}_1 \cup \mathcal{B}_2$ where $x_1, \dots, x_n \in \mathcal{B}_1$ and $y_1, \dots, y_m \in \mathcal{B}_2$ (note that we could have either $n = 0$ or $m = 0$ here). Consider a linear combination $c_1 x_1 + \cdots + c_n x_n + d_1 y_1 + \cdots + d_m y_m = 0$; we need to show that $c_1 = \cdots = c_n = d_1 = \cdots = d_m = 0$. The above equation implies that

$$c_1 x_1 + \cdots + c_n x_n = -d_1 y_1 - \cdots - d_m y_m.$$

Since $x_1, \dots, x_n \in \mathcal{B}_1$, we know that the left-hand side is an element of S_1. Since $y_1, \dots, y_m \in \mathcal{B}_2$, we know that the right-hand side is an element of S_2. As such, we know that $c_1 x_1 + \cdots + c_n x_n$ and $-d_1 y_1 - \cdots - d_m y_m$ are both elements of $S_1 \cap S_2$. However, $S_1 \cap S_2 = \{0\}$, so

$$c_1 x_1 + \cdots + c_n x_n = 0$$
$$-d_1 y_1 - \cdots - d_m y_m = 0$$

Finally, for each $i \in \{1, 2\}$, $\mathcal{B}_i$ is a basis for S_i, so it is linearly independent. This implies that $c_1 = \cdots = c_n = 0$ and $d_1 = \cdots = d_m = 0$. □

(b) Show that if $S_1, S_2 \leq V$ are finite-dimensional and $S_1 \cap S_2 = \{0\}$, then $\dim(S_1 + S_2) = \dim S_1 + \dim S_2$.

Solution: By part (a), if $\mathcal{B}_i$ is a basis for S_i, then $\mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $S_1 + S_2$. Furthermore, $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$, so

$$\dim(S_1 + S_2) = |\mathcal{B}_1 \cup \mathcal{B}_2| = |\mathcal{B}_1| + |\mathcal{B}_2| = \dim(S_1) + \dim(S_2).$$

□

(c) Use the Steinitz exchange lemma to prove that if $\dim V < \infty$ and $X \subseteq V$ is linearly independent, then there is a basis $\mathcal{B}$ for V with $X \subseteq \mathcal{B}$. This is known as the Basis Extension Lemma.

Solution: Suppose $\dim V = n$ and let $\mathcal{B}' = \{b_1, \ldots, b_n\}$ be any basis for V. Suppose also that $X = \{x_1, \ldots, x_m\}$. Since X is linearly independent and $\mathcal{B}'$ is spanning, the Steinitz exchange lemma tells us that $m \leq n$ and we can re-label the vectors in $\mathcal{B}'$ so that $\mathcal{B} = \{x_1, \ldots, x_m, b_{m+1}, \ldots, b_n\}$ is spanning. Now, $\mathcal{B}$ is a set of n vectors which spans V, so $\mathcal{B}$ must be a basis for V. □

(d) **Challenge:** Show that if $S_1, S_2 \leq V$ are finite-dimensional, then $\dim(S_1 + S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2)$.

Solution: Since S_1, S_2 are finite dimensional, so is $S_1 \cap S_2$. Let $\mathcal{B}$ be a basis for $S_1 \cap S_2$.
By the basis extension lemma, we can find $\mathcal{B}_i \supseteq \mathcal{B}$ such that $\mathcal{B}_i$ is a basis for S_i. Set $\mathcal{B}_2' = \mathcal{B}_2 \setminus \mathcal{B}$ and set $S_2' = \operatorname{span} \mathcal{B}_2'$. We notice that $\mathcal{B}_2'$ is a basis for S_2', so $\dim S_2' = \dim S_2 - \dim(S_1 \cap S_2)$. Furthermore, $S_1 + S_2 = S_1 + S_2'$.
Finally, we claim that $S_1 \cap S_2' = \{0\}$. If not, then there is some non-zero $v \in S_1 \cap S_2'$. Now, $S_1 \cap S_2' \subseteq S_1 \cap S_2$ and also $S_1 \cap S_2' \subseteq S_2'$. Thus, v can be written as a linear combination of vectors in $\mathcal{B}$ and can also be written as a linear combination of vectors in $\mathcal{B}_2'$. Since $v \neq 0$, this implies that v can be written as a linear combination of vectors in $\mathcal{B}_2$ in at least two different ways, contradicting the fact that $\mathcal{B}_2$ is linearly independent.
Thus, $S_1 \cap S_2' = \{0\}$, so we apply part (b) to conclude that

$$\dim(S_1 + S_2) = \dim(S_1 + S_2') = \dim S_1 + \dim S_2' = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2).$$

□

(6) Show that if $S \leq V$ is a proper subspace (that is $S \neq V$), then $\operatorname{span}(S^C) = V$.

Solution: It suffices to show that $S \subseteq \operatorname{span}(S^C)$ since we already know that $S^C \subseteq \operatorname{span}(S^C)$ and $S \cup S^C = V$.

Since $S \neq V$, we know that $S^C \neq \varnothing$. Furthermore, $0 \in S$, so $0 \notin S^C$, meaning that there is some non-zero $v \in S^C$

Fix any $s \in S$; we need to show that $s \in \operatorname{span}(S^C)$. Consider $s + v$; if $s + v \in S$, then since $S \leq V$, we would have $v = (s + v) - s \in S$, which isn't true. Therefore, $s + v \in S^C$, so $s = (s + v) - v \in \operatorname{span}(S^C)$ as needed. □

(7) Recall that a function $f \colon X \to Y$ is called an *injection* (one-to-one) $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$; and is called a *surjection* (onto) if for every $y \in Y$, there is $x \in X$ with $f(x) = y$.

For this exercise, pretend that the first week of class did not happen, i.e. we have never seen a pivot before, we don't know what an inverse matrix is, etc. That is, answer these questions using only basic matrix operations and facts about subspaces.

(a) Let $A \in \mathbb{R}^{m \times n}$ and consider A as a function $A \colon \mathbb{R}^n \to \mathbb{R}^m$. Show that A is a surjection if and only if $\operatorname{rank} A = m$.

Solution: Since $\operatorname{Col} A = \{\vec{b} \in \mathbb{R}^m : A\vec{x} = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^n\}$, we see that A is a surjection if and only if $\operatorname{Col} A = \mathbb{R}^m$. Since $\operatorname{Col} A \leq \mathbb{R}^m$ always and $\operatorname{rank} A = \dim \operatorname{Col} A$, we see that A is a surjection if and only if $\operatorname{rank} A = m$. □

(b) Show that A is an injection if and only if $\operatorname{Nul} A = \{\vec{0}\}$.

Solution: First suppose that A is an injection, so, in particular, the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$; i.e. $\operatorname{Nul} A = \{\vec{0}\}$.

Now suppose that $\operatorname{Nul} A = \{\vec{0}\}$. Suppose that $\vec{x}, \vec{y} \in \mathbb{R}^n$ has $A\vec{x} = A\vec{y}$. This happens if and only if $A(\vec{x} - \vec{y}) = \vec{0}$, or equivalently, $\vec{x} - \vec{y} \in \operatorname{Nul} A$. Since $\operatorname{Nul} A = \{\vec{0}\}$, this means that $\vec{x} - \vec{y} = \vec{0} \implies \vec{x} = \vec{y}$, i.e. A is an injection. $\square$

(c) Let $A \in \mathbb{R}^{n \times n}$. Show that A is an injection if and only if A is a surjection.

Solution: By the rank–nullity theorem, we know that $\dim \operatorname{Nul} A + \operatorname{rank} A = n$ here.

If A is an injection, then $\dim \operatorname{Nul} A = 0$ by part (b), which means that $\operatorname{rank} A = n$, so A is a surjection by part (a). Similarly, if A is a surjection, then $\operatorname{rank} A = n$ by part (a), so $\dim \operatorname{Nul} A = 0$, so A is an injection by part (b). $\square$

(8) Does there exist a matrix $A \in \mathbb{R}^{3435 \times 3435}$ with $\operatorname{Nul} A = \operatorname{Col} A$?

Solution: No. The rank–nullity theorem tells us that $\dim \operatorname{Nul} A + \dim \operatorname{Col} A = 3435$ in this case. But if $\dim \operatorname{Nul} A = \dim \operatorname{Col} A$, then $\dim \operatorname{Nul} A + \dim \operatorname{Col} A$ is an even number, which 3435 is not. $\square$