Justify all answers! I recommend doing these questions out of order and focus first on questions with which you are less comfortable.

Please let me know if I've made any mistakes in my solutions.

(1) Solve the following linear equations:

(a) $\begin{bmatrix} 2 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & -1 & -1 \\ 4 & 1 & 2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ -1 \\ 0 \\ 9 \end{bmatrix}.$

Solution:

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 2 & 6 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 0 & -1 & -1 & 0 \\ 4 & 1 & 2 & 1 & 9 \end{array}\right] \overset{\frac{1}{2}\cdot r_1,\ r_3-3r_1,\ r_4-4r_1}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -4 & -4 & -9 \\ 0 & 1 & -2 & -3 & -3 \end{array}\right] \overset{r_4-r_2}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -4 & -4 & -9 \\ 0 & 0 & -2 & -2 & -2 \end{array}\right]$$

$$\overset{r_4-\frac{1}{2}r_3}{\rightsquigarrow} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -4 & -4 & -9 \\ 0 & 0 & 0 & 0 & \frac{5}{2} \end{array}\right]$$

There is no solution. □

(b) $\begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \vec{x} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix}.$

Solution:

$$\left[\begin{array}{ccc|c} 3 & 0 & 1 & 7 \\ 1 & 1 & 3 & 4 \\ 1 & 2 & -5 & -1 \end{array}\right] \overset{r_2\to r_1\to r_3\to r_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 1 & 2 & -5 & -1 \\ 3 & 0 & 1 & 7 \end{array}\right] \overset{r_2-r_1,\ r_3-3r_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -8 & -5 \\ 0 & -3 & -8 & -5 \end{array}\right]$$

$$\overset{r_3+3r_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & -32 & -20 \end{array}\right]$$

Therefore, $\vec{x} = \frac{1}{8}\begin{bmatrix} 17 \\ 0 \\ 5 \end{bmatrix}$. □

(2) Fix $A \in \mathbb{R}^{m\times n}$.

(a) When does $A\vec{x} = \vec{0}$ have a unique solution?

Solution: This happens precisely when A has full column rank. □

(b) In general, what are the possible number of solutions to $A\vec{x} = \vec{b}$? (e.g. can there be exactly 3 solutions?)

Solution: There are either no solutions, one solution, or infinitely many solutions.

This follows from the fact that $A\vec{x} = \vec{0}$ has either one solution or infinitely many, and if $A\vec{x} = \vec{b}$ has a solution, call one $\vec{x}_p$, then the full set of solutions is $\{\vec{x}_p + \vec{x}_h : A\vec{x}_h = \vec{0}\}$. □

(c) If there is some $\vec{b} \in \mathbb{R}^m$ for which $A\vec{x} = \vec{b}$ has at least two solutions, how many solutions does $A\vec{x} = \vec{0}$ have?

Solution: Let $\vec{x}_p$ be one of the solutions to $A\vec{x} = \vec{b}$. Then the full solution set is $\{\vec{x}_p + \vec{x}_h : A\vec{x}_h = \vec{0}\}$. Since this set has at least two elements, we see that there must be at least two solutions to $A\vec{x} = \vec{0}$, and thus there must in fact be infinitely many. □

(d) Show that if $m \neq n$, then either there is some $\vec{b} \in \mathbb{R}^m$ for which $A\vec{x} = \vec{b}$ has no solution, or there is some $\vec{b} \in \mathbb{R}^m$ for which $A\vec{x} = \vec{b}$ has infinitely many solutions.

Solution: Since $\operatorname{rank} A \leq \min\{m, n\}$ and $m \neq n$, we know that either A does not have full row rank, or does not have full column rank. This then implies one of the two conclusions. □

(e) Suppose that $m = n$ and there is some $\vec{b} \in \mathbb{R}^n$ for which $A\vec{x} = \vec{b}$ has no solution. How many solutions does $A\vec{x} = \vec{0}$ have?

Solution: If there is some $\vec{b}$ for which $A\vec{x} = \vec{b}$ has no solution, then A cannot have full row rank. Since A is square, this means also that A does not have full column rank, so $A\vec{x} = \vec{0}$ must have a non-trivial solution. Therefore, $A\vec{x} = \vec{0}$ has infinitely many solutions. □

(f) Suppose that $m = n$ and that $A\vec{x} = \vec{0}$ has only the trivial solution. How many solutions does $A^2\vec{x} = \vec{0}$ have? Prove your claim without using any facts about rank or inverses.

Solution: Suppose that $A^2\vec{x} = \vec{0}$ were to have a non-trivial solution; call it $\vec{y}$. We claim that $A\vec{x} = \vec{0}$ also has a non-trivial solution. If $A\vec{y} = \vec{0}$, then we're done, so suppose that $A\vec{y} \neq \vec{0}$. But then $\vec{0} = A^2\vec{y} = A(A\vec{y})$, so $A\vec{y}$ is a non-trivial solution to $A\vec{x} = \vec{0}$. □

(3) For $\vec{u}, \vec{v} \in \mathbb{R}^n$ where $\vec{u} \neq \vec{0}$, the line in the direction of $\vec{u}$ which passes through the point $\vec{v}$ is the set of points $\{\vec{v} + t\vec{u} : t \in \mathbb{R}\}$. Suppose we have two lines ℓ_1, ℓ_2 in $\mathbb{R}^n$.

How can we use a linear equation to determine whether or not ℓ_1 and ℓ_2 intersect?

Solution: Suppose that $\ell_i = \{\vec{v}_i + t\vec{u}_i : t \in \mathbb{R}\}$. Note that ℓ_1 and ℓ_2 intersect if and only if there are $t, s \in \mathbb{R}$ for which $\vec{v}_1 + t\vec{u}_1 = \vec{v}_2 + s\vec{u}_2$. In other words, ℓ_1 and ℓ_2 intersect if and only if there is a solution to the linear equation

$$\begin{bmatrix} \vec{u}_1 & -\vec{u}_2 \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \vec{v}_2 - \vec{v}_1.$$

□

(4) For $\vec{u}, \vec{v} \in \mathbb{R}^n$ where $\vec{u} \neq \vec{0}$, the hyperplane perpendicular to $\vec{u}$ which passes through the point $\vec{v}$ is the set of points $\{\vec{x} \in \mathbb{R}^n : \vec{u}^T(\vec{x} - \vec{v}) = 0\}$.

For hyperplanes $H_1 \neq H_2$ in $\mathbb{R}^n$, when do H_1 and H_2 intersect?

Solution: Suppose that $H_i = \{\vec{x} \in \mathbb{R}^n : \vec{u}_i^T(\vec{x} - \vec{v}_i) = 0\}$ and notice that $H_i = \{\vec{x} \in \mathbb{R}^n : \vec{u}_i^T\vec{x} = c_i\}$ where $c_i = \vec{u}_i^T\vec{v}_i$.

Therefore, H_1 and H_2 intersect if and only if there is a solution to

$$\begin{bmatrix} \vec{u}_1^T \\ \vec{u}_2^T \end{bmatrix} \vec{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

By relabeling if necessary (i.e. switching the "1" and "2"), the REF augmented matrix for this equation will look like

$$\left[\begin{array}{c|c} \vec{u}_1^T & c_1 \\ \vec{u}_2^T - \alpha\vec{u}_1^T & c_2 - \alpha c_1 \end{array}\right]$$

for some $\alpha \in \mathbb{R}$.

Firstly, there is no solution if and only if $\vec{u}_2^T - \alpha\vec{u}_1^T = \vec{0}^T$, but $c_2 - \alpha c_1 \neq 0$. In other words, H_1 and H_2 are parallel, but not equal.

Next, if both $\vec{u}_2^T - \alpha\vec{u}_1^T = \vec{0}^T$ and $c_2 - \alpha c_1 = 0$, then H_1 and H_2 are actually the same hyperplane.

Otherwise, $\vec{u}_2^T - \alpha\vec{u}_1^T \neq \vec{0}^T$, so we know that the matrix in this equation has rank 2, which implies that there will always be a solution.

Putting this together, $H_1 \neq H_2$ intersect if and only if they are not parallel. □

(5) Let $A, B \in \mathbb{R}^{n\times n}$.

(a) Show that A is non-singular if and only if A^T is non-singular.

Solution: We find that $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$, and so $(A^T)^{-1} = (A^{-1})^T$. □

(b) Show that if A is non-singular, then so is A^{-1}.

Solution: We know that $A^{-1}A = I_n$, so in fact $(A^{-1})^{-1} = A$. □

(c) Show that if both A and B are non-singular, then so is AB.

Solution: We verify $(B^{-1}A^{-1})(AB) = B^{-1}I_nB = I_n$, so $(AB)^{-1} = B^{-1}A^{-1}$. □

(d) Show that if AB is non-singular, then so are both A and B.

Solution: We know that $(AB)^{-1}(AB) = (AB)(AB)^{-1} = I_n$. Grouping terms together differently, we see that $\left[(AB)^{-1}A\right]B = I_n$ and $A\left[B(AB)^{-1}\right] = I_n$. Thus, both A and B have inverses. □

(e) Suppose that A is non-singular. Show that $AB = BA$ if and only if $A^{-1}B = BA^{-1}$.

Solution: By multiplying the equation $AB = BA$ on both the left and right by A^{-1} we find

$$A^{-1}ABA^{-1} = A^{-1}BAA^{-1} \implies BA^{-1} = A^{-1}B.$$

Since $(A^{-1})^{-1} = A$, this shows the other direction as well. □

(f) Suppose that A is non-singular and B is row-reducible to A. Must B be non-singular?

Solution: Yes. Since B is row-reducible to A, there is a matrix $C \in \mathbb{R}^{n\times n}$ such that $CB = A$. Therefore $A^{-1}CB = I_n$, so B is non-singular with $B^{-1} = A^{-1}C$. □

(g) Suppose that $A^4 = O_n$ where O_n is the $n \times n$ zero matrix. Show that A must be singular, yet $I_n - A$ is non-singular with $(I_n - A)^{-1} = I_n + A + A^2 + A^3$.

Solution: If A were non-singular, then $A^4(A^{-1})^4 = AAAAA^{-1}A^{-1}A^{-1}A^{-1} = I_n^4 = I_n$, so A^4 must be non-singular as well. But $A^4 = O_n$ which is singular; a contradiction.
To show that $(I_n - A)^{-1} = I_n + A + A^2 + A^3$, we simply verify

$$(I_n - A)(I_n + A + A^2 + A^3) = I_n + A + A^2 + A^3 - A - A^2 - A^3 - A^4 = I_n - A^4 = I_n.$$

□