Here we use Zorn's Lemma to prove that if V is any vector space over a field $\mathbb{F}$, then V has a basis. We've shown in class that any maximal linearly independent set is a basis, so we just need to show that one of these exists. Lucky for us, Zorn's Lemma is perfect for finding maximal elements!

Proof. Let P be the poset whose elements are linearly independent subsets of P where $X \preceq Y$ whenever $X \subseteq Y$. Notice that P is non-empty since $\varnothing$ is always linearly independent.

Let $\mathcal{C}$ be a chain in P, we need to show that $\mathcal{C}$ has an upper bound in P. Set $C^* = \bigcup_{X \in \mathcal{C}} X$. Certainly $X \subseteq C^*$ for every $X \in \mathcal{C}$, so C^* would be an upper bound for $\mathcal{C}$ if $C^* \in P$; that is, if C^* is linearly independent.

To show that this is the case, consider a linear combination $c_1x_1 + \cdots + c_nx_n = 0$ where $x_1, \ldots, x_n \in C^*$ and $c_1, \ldots, c_n \in \mathbb{F}$. Since $\mathcal{C}$ is a chain and there are only finitely many x_i's, we can find some $X^* \in \mathcal{C}$ for which $x_1, \ldots, x_n \in X^*$. Now, X^* is linearly independent, so since $c_1x_1 + \cdots + c_nx_n = 0$, we know that $c_1 = \cdots = c_n = 0$. Therefore $C^* \in P$, so C^* is an upper bound for the chain $\mathcal{C}$.

We now apply Zorn's Lemma to P to get the existence of a maximal element X^* of P. By definition, X^* is a maximal linearly independent subset of V, so X^* is a basis for V. $\square$