Theorem 1. *If G is an n-vertex graph, then $\chi(G) \cdot \chi(\overline{G}) \geq n$.*

There are many proofs of this fact; here are two:

Proof. We know that $\omega(\overline{G}) = \alpha(G)$, so by applying our two lower bounds on the chromatic number, we have

$$\chi(G) \cdot \chi(\overline{G}) \geq \frac{n}{\alpha(G)} \cdot \omega(\overline{G}) = n. \qquad \square$$

Proof. Let $f_1 \colon V \to [\chi(G)]$ be a proper coloring of G and let $f_2 \colon V \to [\chi(\overline{G})]$ be a proper coloring of $\overline{G}$.

We then define $f \colon V \to [\chi(G)] \times [\chi(\overline{G})]$ by

$$f(x) = \big(f_1(x), f_2(x)\big).$$

Observe that f is a $\chi(G) \cdot \chi(\overline{G})$-coloring. We claim that $f(x) \neq f(y)$ for all $x \neq y \in V$, which will imply that $\chi(G) \cdot \chi(\overline{G}) \geq n$.

Indeed, fix any $x \neq y \in V$.

Case 1: $xy \in E(G)$. Since f_1 is a proper coloring of G, we then have $f_1(x) \neq f_1(y)$ and so $f(x) \neq f(y)$.

Case 2: $xy \notin E(G)$. Then $xy \in E(\overline{G})$ since $x \neq y$. Since f_2 is a proper coloring of $\overline{G}$, we then have $f_2(x) \neq f_2(y)$ and so $f(x) \neq f(y)$. $\square$

While the second proof probably seems more complicated, the idea held therein is much more versatile (wink wink nudge nudge).