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Let G be a graph. A (vertex-)coloring of G is simply a function $f: V(G) \rightarrow C$ where C is some set. The set C is often referred to as a “palette” of colors. Intuitively, we are “coloring” each vertex of G with some color in the “palette” C . Mostly, this just gives us a colloquial way to discuss arbitrary functions on $V(G)$. We (kinda) pecified “vertex-coloring”, since later we will discuss “edge-colorings” which are simply functions from $E(G)$ to some set. In general, when anyone in math mentions a “coloring” of an object, they are literally just discussing a function from said object to an arbitrary set.

For a positive integer t , a t -coloring (or t -vertex-coloring if such a distinction is necessary later) is a coloring where the “palette” has t colors. Since the actual nature of the “palette” doesn’t usually matter, usually, a t -coloring is a function $f: V(G) \rightarrow [t]$. Note that, technically, not every color must be used in a t -coloring. In particular, a t -coloring is also a $(t + 1)$ -coloring, technically.

So far, colorings are simply functions with domain $V(G)$ and have nothing to do with the actual graph G ; let’s fix this. A *proper coloring* of G is a function $f: V(G) \rightarrow C$, where C is some set, such that $xy \in E(G) \implies f(x) \neq f(y)$. That is, no two end-points of an edge receive the same color. It is very rare that we will discuss colorings which are not proper (since we care about the graph, afterall).

Reasonably, a *proper t -coloring* of G is a proper coloring of G which uses at most t colors. Intuitively, if G has a lot of edges, then G requires quite a few colors in order to have a proper coloring. We will see that this is not exactly true, but it’s a reasonable intuition.

Definition 1. Let G be a graph. The chromatic number of G , denoted by $\chi(G)$, is the smallest integer t such that G has a proper t -coloring.

A quick observation:

Observation 2. $\chi(G) = 1$ if and only if G has no edges.

Technically this is only true if G has at least one vertex. Your book assumes that any graph satisfies this (and I generally do as well), but sometimes the “null-graph” (the graph with no vertices) can show up if necessary. Note that the null-graph has chromatic number 0.

A couple more quick observations:

Observation 3. If G is an n -vertex graph, then $\chi(G) \leq n$. Furthermore, $\chi(G) = n$ if and only if $G \cong K_n$.

Observation 4. If H is a subgraph of G , then $\chi(H) \leq \chi(G)$.

From here, we get a simple lower-bound on the chromatic number of a graph in terms of it’s clique number.

Definition 5. The clique-number of a graph G , denoted by $\omega(G)$, is the size of the largest clique contained within G . Note that $\omega(G) = \alpha(\overline{G})$.

By definition, G contains a copy of $K_{\omega(G)}$, which has chromatic number $\omega(G)$, so:

Observation 6. $\chi(G) \geq \omega(G)$.

This bound is far from tight. It is known that there are graphs with $\omega(G) = 2$, yet $\chi(G)$ is arbitrarily large.

Let $f: V(G) \rightarrow C$ be any proper coloring of G . A *color class* of this coloring is the set $f^{-1}(c)$ for any $c \in C$; i.e. the set of vertices which receive color c . Since no two adjacent vertices receive the same color, we observe that every color class is an independent set. In other words, the color classes of a proper coloring $f: V(G) \rightarrow C$ partition $V(G)$ into $|C|$ many independent sets. Conversely, if $V(G) = A_1 \sqcup \cdots \sqcup A_t$ is a partition wherein each A_i is an independent set of G , then we can define a proper coloring $f: V(G) \rightarrow [t]$ via $f(v) = i$ iff $v \in A_i$. Thus, provided we don't care about the actual nature of the colors, proper colorings are equivalent to partitions into independent sets.

Observation 7. $\chi(G)$ is the smallest integer t such that $V(G)$ can be partitioned into t independent sets. In other words, $\chi(G)$ is the smallest integer t for which G is t -partite.

In particular, $\chi(G) = 2$ if and only if G is bipartite and has at least one edge.

This observation leads to the following lower bound on the chromatic number:

Theorem 8. For any n -vertex graph G ,

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

Proof. By definition, we can partition $V(G) = A_1 \sqcup \cdots \sqcup A_{\chi(G)}$ where each A_i is an independent set in G . Thus, $|A_i| \leq \alpha(G)$ since $\alpha(G)$ is the size of a largest independent set in G . We therefore bound

$$n = \sum_{i=1}^{\chi(G)} |A_i| \leq \sum_{i=1}^{\chi(G)} \alpha(G) = \chi(G)\alpha(G) \implies \chi(G) \geq \frac{n}{\alpha(G)}. \quad \square$$

Again, this bound is far from tight in general.

Let's now discuss an upper bound. Recall the notion of degeneracy from DS3.5.

Definition 9. For a graph G , the degeneracy of G is defined to be

$$d(G) = \max\{\delta(H) : H \text{ is a subgraph of } G\}.$$

Certainly $d(G) \leq \Delta(G)$ always. Also, we discussed that $d(G) \leq 1$ if and only if G is a forest, so degeneracy can sometimes be much smaller than the max degree.

Theorem 10. $\chi(G) \leq d(G) + 1$. In particular, $\chi(G) \leq \Delta(G) + 1$.

We give two proofs of this fact.

Proof. We prove the claim by induction on $n = |V(G)|$.

If $n = 1$, then $d(G) = 0$ and $\chi(G) = 1$.

Now consider $n \geq 2$ and let $v \in V(G)$ be a vertex with $\deg v = d(G)$; note that $\deg v \leq d(G)$. Set $H = G - v$. Since H is a subgraph of G , we have $d(H) \leq d(G)$ and so the induction hypothesis tells us that $\chi(H) \leq d(H) + 1 \leq d(G) + 1$. Thus, suppose that $f: V(H) \rightarrow C$ is a proper coloring of H where C is some set with $|C| = d(G) + 1$ (note that, perhaps, many colors could be unused). We show that we can extend f to a proper coloring of G . Indeed, since $|C| = d(G) + 1$ and $\deg v \leq d(G)$, there is some color $c \in C$ which is unused by the neighbors of v . Thus, setting $f(v) = c$, we know that f is now a proper coloring of G using at most $d(G) + 1$ many colors and so $\chi(G) \leq d(G) + 1$. \square

Proof. In DS3.5.2, you showed that there is an ordering $V(G) = \{v_1, \dots, v_n\}$ so that $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d(G)$ for all $i \in [n]$. We define a proper coloring $f: V(G) \rightarrow [d(G) + 1]$ greedily based on this ordering.

Start by setting $f(v_1) = 1$. For $i \in \{2, \dots, n\}$, assuming that f has already been defined on $\{v_1, \dots, v_{i-1}\}$, define $f(v_i)$ to be the smallest color not used by the already-colored neighbors of v_i (that is, the smallest color not used in $N(v_i) \cap \{v_1, \dots, v_{i-1}\}$). Since there are $d(G) + 1$ many available colors and $|N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq d(G)$, this is always possible.

The resulting coloring f is a proper coloring. Indeed, if $v_i v_j \in E(G)$ where $i < j$, then $v_i \in N(v_j) \cap \{v_1, \dots, v_{j-1}\}$ and so $f(v_i) \neq f(v_j)$ by construction.

Thus, since f uses at most $d(G) + 1$ many colors, we've shown that $\chi(G) \leq d(G) + 1$. \square

Degeneracy can oftentimes be difficult to work with theoretically, so it's usually best to just keep the bound $\chi(G) \leq \Delta(G) + 1$ in mind.

Let's finish today by proving a general upper bound on the chromatic number of a triangle-free graph in terms of its number of vertices. A graph is triangle-free if it has no copy of K_3 ; equivalently $\omega(G) \leq 2$.

Theorem 11. *If G is an n -vertex, triangle-free graph, then $\chi(G) \leq \sqrt{2n}$.*

Proof. We prove the claim by induction on n .

If $n = 1$, then $\chi(G) = 1 \leq \sqrt{2}$, so suppose that $n \geq 2$.

If $\Delta(G) \leq \sqrt{2n} - 1$, then we would have $\chi(G) \leq \Delta(G) + 1 \leq \sqrt{2n}$ as needed; thus we may suppose that $\Delta(G) > \sqrt{2n} - 1$. Fix any vertex $v \in V(G)$ with $\deg v > \sqrt{2n} - 1$.

Since G is triangle-free, we know that $N(v)$ must be an independent set in G . If $V(G) = N(v) \cup \{v\}$, then G is bipartite and so $\chi(G) \leq 2 \leq \sqrt{2n}$ since $n \geq 2$, so we may suppose that this is not the case.

Set $H = G - (N(v) \cup \{v\})$ (deleting vertices). We claim that $\chi(G) \leq 1 + \chi(H)$.

Let $A_1, \dots, A_{\chi(H)}$ be a partition of $V(H)$ into $\chi(H)$ -many independent sets; note that $\chi(H) \geq 1$ since H has some vertices. H is an induced subgraph of G and so each A_i is also an independent set in G . Now, v has no neighbors in H and so $A_1 \cup \{v\}$ is also an independent set. Finally, $N(v)$ is an independent set, and so $A_1 \cup \{v\}, A_2, \dots, A_{\chi(H)}, N(v)$ is a partition of $V(G)$ into $\chi(H) + 1$ many independent sets and so $\chi(G) \leq 1 + \chi(H)$ as claimed.

Now, H is a subgraph of G and so is also triangle-free. Furthermore, H has $n - \deg v - 1 < n$ many vertices, so the induction hypothesis tells us that

$$\chi(H) \leq \sqrt{2(n - \deg v - 1)} < \sqrt{2(n - \sqrt{2n})},$$

since $\deg v > \sqrt{2n} - 1$. Therefore, since $n \geq 2$,

$$\chi(G) \leq 1 + \chi(H) < 1 + \sqrt{2(n - \sqrt{2n})} < 1 + \sqrt{2n - 2\sqrt{2n} + 1} = 1 + \sqrt{(\sqrt{2n} - 1)^2} = \sqrt{2n}. \quad \square$$

Note that our proof actually showed that $\chi(G) < \sqrt{2n}$, unless $G \cong K_2$. It turns out that the bound in Theorem 11 is not too far from the truth. It is known that if n is large, then

$$C_1 \sqrt{\frac{n}{\log n}} \leq \max_{G \text{ triangle-free on } n \text{ vertices}} \chi(G) \leq C_2 \sqrt{\frac{n}{\log n}},$$

for some constants C_1, C_2 .