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Here is a proof of Menger's theorem which is slightly different from that in the book. The main workhorse of the proof is Lemma 1 and it will be our friend later in the course as well :)

Let G be a graph and fix any $A, B \subseteq V(G)$ with A, B non-empty (note that A and B may intersect). An A - B path is a path (v_0, \dots, v_k) in G such that $v_0 \in A$, $v_k \in B$ and none of the internal vertices v_1, \dots, v_{k-1} live in either A or B . Note that if $a \in A \cap B$, then (a) is an A - B path.

An A - B separator is a set $S \subseteq V(G)$ such that $G - S$ contains no $(A \setminus S)$ - $(B \setminus S)$ path.¹ Note that $S = A$ and $S = B$ are trivially A - B separators; in particular, A - B separators exist. Let $\kappa_G(A, B)$ denote the size of a minimum A - B separator in G .

Observe that S is an A - B separator if and only if every A - B path in G contains a vertex from S . In particular, if P_1, \dots, P_k are vertex-disjoint A - B paths, then $k \leq \kappa_G(A, B)$. Let $p_G(A, B)$ denote the maximum number of vertex-disjoint A - B paths in G .

Lemma 1. *Let G be a graph and fix any $A, B \subseteq V(G)$ with A, B non-empty. Then $p_G(A, B) = \kappa_G(A, B)$.*

Often, a collection of vertex-disjoint A - B paths in G is called an A - B connector. With this language, the lemma states that any maximum A - B connector has the same size as any minimum A - B separator.

Proof. We have already noted that $p_G(A, B) \leq \kappa_G(A, B)$, so we must prove that $p_G(A, B) \geq \kappa_G(A, B)$. We prove this by induction on $|E(G)|$.

For the base case, consider when $|E(G)| = 0$. Then every A - B path has the form (x) for some $x \in A \cap B$ (should such an x exist). Thus, $p_G(A, B) = |A \cap B| = \kappa_G(A, B)$.

Suppose now that $|E(G)| \geq 1$ and fix any $e \in E(G)$. Set $k = \kappa_G(A, B)$; we must find k vertex-disjoint A - B paths in G in order to show that $p_G(A, B) \geq k$. Consider the graph $G - e$. It must be the case that $\kappa_{G-e}(A, B) \in \{k, k - 1\}$ (why? It's the same reasoning as in HW7.3.2). If $\kappa_{G-e}(A, B) = k$, then $p_{G-e}(A, B) = k$ by the induction hypothesis and so there are k vertex-disjoint A - B paths in $G - e$. Each of these paths is also a path in G and so $p_G(A, B) \geq k$ as needed.

Thus, suppose that $\kappa_{G-e}(A, B) = k - 1$. Let S be a minimum A - B separator in $G - e$, so $|S| = k - 1$. By assumption, $\kappa_G(A, B) = k > |S|$ and so there is an A - B path in $G - S$. Since these paths don't exist in $(G - e) - S$, it must be the case that each of them use the edge e . Let P be any such path and label $e = v_1v_2$ such that P uses the vertex v_1 before the vertex v_2 . Observe that every A - $\{v_2\}$ path in $G - S$ and every $\{v_1\}$ - B path in $G - S$ uses the edge e or else we could avoid e all-together.

Now, we know that $v_1, v_2 \notin S$. For $i \in \{1, 2\}$, set $S_i = S \cup \{v_i\}$; then $|S_i| = k$. Additionally, both S_1 and S_2 are A - B separators in G since every A - B path in $G - S$ used the edge $e = v_1v_2$.

Let T_1 be a minimum A - S_1 separator in $G - e$ and let T_2 be a minimum S_2 - B separator in $G - e$. We claim that T_1, T_2 are both A - B separators in G . Indeed, suppose for the sake of contradiction that there were an A - B path in $G - T_1$; call it P . Since S_1 is an A - B separator, the path P must contain a vertex from S_1 , let P' be the sub-path of P which ends at the first vertex encountered in S_1 . Then P' is an A - S_1 path in $G - T_1$, so since T_1 is an A - S_1 separator in $G - e$, this means that P' uses the edge e . Since $v_1 \in S_1$, the only way that this is possible is if P' ends in (v_2, v_1) . But then P' sans the last vertex v_1 is an A - $\{v_2\}$ path in $G - S_1 = G - (S \cup \{v_1\})$ contradicting the fact

¹The book also defines a notion of a separator which is a little bit different than this, so be careful.

that any $A-\{v_2\}$ path in $G - S$ must use the edge e . A symmetric argument works to show that T_2 is also an $A-B$ separator.

Since T_1, T_2 are $A-B$ separators in G , we must have $|T_i| \geq \kappa_G(A, B) = k$. Therefore, $\kappa_{G-e}(A, S_1) = |T_1| \geq k$ and $\kappa_{G-e}(S_2, B) = |T_2| \geq k$. By the induction hypothesis, we therefore know that

$$p_{G-e}(A, S_1) = \kappa_{G-e}(A, S_1) \geq k \quad \text{and} \quad p_{G-e}(S_2, B) = \kappa_{G-e}(S_2, B) \geq k.$$

Thus, there are at least k vertex-disjoint $A-S_1$ paths in $G - e$ and at least k vertex-disjoint S_2-B paths in $G - e$. Since $|S_1| = |S_2| = k$, there must actually be exactly k of each. Furthermore, by labeling $S = \{s_1, \dots, s_{k-1}\}$, this same fact means that we can find vertex-disjoint $A-S_1$ paths P_1, \dots, P_k such that the last vertex of P_k is v_1 and, for each $i \in [k-1]$, the last vertex of P_i is s_i . Similarly, we can find vertex-disjoint S_2-B paths P'_1, \dots, P'_k such that the first vertex of P'_k is v_2 and, for each $i \in [k-1]$, the first vertex of P'_i is s_i . For each $i \in [k-1]$ note that concatenating P_i and P'_i is an $A-B$ path in G . Also, since $v_1 v_2 \in E(G)$, concatenating P_k and P'_k is also an $A-B$ path in G . After all this work, we see that these k concatenated paths are vertex-disjoint and so $p_G(A, B) \geq k$ as needed. \square

After all that work, we can finally prove Menger's Theorem. Recall that paths are said to be internally-disjoint if they share no vertices beyond their end-points.

Theorem 2 (Menger's Theorem for vertex-connectivity). *Let G be a graph on at least two vertices. G is k -connected (i.e. $\kappa(G) \geq k$) if and only if there are at least k internally-disjoint $u-v$ paths for every $u \neq v \in V(G)$.*

Proof. (\Leftarrow) We prove the contrapositive: suppose that G is not k -connected, so $\kappa(G) \leq k-1$. We need to show that there are some $u \neq v \in V(G)$ for which there are at most $k-1$ many internally-disjoint $u-v$ paths in G .

If G is a clique, then $G \cong K_n$ for some $n \leq k$ since $\kappa(K_n) = n-1$. Consider any $u \neq v \in V(G)$. Since $|V(G) \setminus \{u, v\}| = n-2$ there are at most $n-2$ internally-disjoint $u-v$ paths which use some vertex other than u and v . Along with the path (u, v) , this yields at most $n-1 \leq k-1$ many internally-disjoint $u-v$ paths.

Now suppose that G is not a clique; thus there is some $U \subseteq V(G)$ with $|U| = \kappa(G) \leq k-1$ such that $G - U$ is disconnected. Consider any u, v in different connected components of $G - U$. Then every $u-v$ path in G must use some vertex in U and hence there are at most $|U| \leq k-1$ many internally-disjoint $u-v$ paths in G .

(\Rightarrow) We proceed by induction on $|E(G)|$. The claim is immediate if $|E(G)| = 0$ since then $\kappa(G) = 0$; thus, suppose that $|E(G)| \geq 1$. Fix any $u \neq v \in V(G)$; we must find at least $\kappa(G)$ many internally-disjoint $u-v$ paths in G .

Case 1: $uv \in E(G)$. By HW7.3.2, we know that $\kappa(G - uv) \geq \kappa(G) - 1$. Thus, the induction hypothesis allows us to conclude that there are at least $\kappa(G - uv) \geq \kappa(G) - 1$ many internally-disjoint $u-v$ paths in $G - uv$. These paths along with the path (u, v) is then a collection of at least $\kappa(G)$ many internally-disjoint $u-v$ paths in G .

Case 2: $uv \notin E(G)$. Let $S \subseteq V(G - \{u, v\}) = V(G) \setminus \{u, v\}$ be a minimum $N(u)-N(v)$ separator in $G - \{u, v\}$, so $|S| = \kappa_{G-\{u,v\}}(N(u), N(v))$. We claim that u and v are in different connected components of $G - S$. Indeed, since $uv \notin E(G)$ every $u-v$ path in G contains an $N(u)-N(v)$ path. All of these are destroyed upon deleting S and so there is no $u-v$ path in $G - S$. In particular, $G - S$ is disconnected and so $|S| \geq \kappa(G)$. Thus, appealing to Lemma 1, we have

$$p_{G-\{u,v\}}(N(u), N(v)) = \kappa_{G-\{u,v\}}(N(u), N(v)) \geq \kappa(G).$$

In particular, there are at least $\kappa(G)$ many vertex-disjoint $N(u)$ - $N(v)$ paths in $G - \{u, v\}$. Appending u to the start and v to the end of each of these paths then yields at least $\kappa(G)$ many internally-disjoint u - v paths in G . \square

There is also an edge-connectivity version of Menger's Theorem.

Theorem 3 (Menger's Theorem for edge-connectivity). *Let G be a graph on at least two vertices. G is k -edge-connected (i.e. $\lambda(G) \geq k$) if and only if there are at least k edge-disjoint u - v paths for every $u \neq v \in V(G)$.*

We will need an intermediate lemma about line graphs.

Lemma 4. *Let G be a graph and fix any $u \neq v \in V(G)$. If (e_0, \dots, e_k) is a path in the line graph $L(G)$ with $u \in e_0$ and $v \in e_k$, then there is a u - v path in G using only edges from $\{e_0, \dots, e_k\}$.*

Proof. Consider the spanning subgraph H of G which has only the edges e_0, \dots, e_k . We claim that H contains a u - v path which will verify the claim.

Consider any partition $V(H) = V(G) = A \sqcup B$ with $u \in A$ and $v \in B$; we must show that H has an edge between A and B . Now, let $i \in \{0, \dots, k\}$ be the smallest index such that $e_i \cap B \neq \emptyset$. We know that i exists since $v \in e_k \cap B$. We claim that $e_i \cap A \neq \emptyset$ as well which will conclude the proof. If $i = 0$, then $u \in e_i \cap A \implies e_i \cap A \neq \emptyset$, so suppose that $i \geq 1$. Then, since e_{i-1} and e_i are adjacent in $L(G)$, we must have $e_{i-1} \cap e_i \neq \emptyset$. By the definition of i , we know that $e_{i-1} \subseteq A$ and so the common vertex of e_{i-1} and e_i lives in A ; thus $e_i \cap A \neq \emptyset$. \square

Proof of Theorem 3. Note that $G \not\cong K_1$ since it has at least two vertices; thus we don't need to worry about any edge-cases here.

(\Leftarrow) We prove the contrapositive: suppose that G is not k -edge-connected, so $\lambda(G) \leq k - 1$. Then we can find an edge-cut $S \subseteq E(G)$ with $|S| = \lambda(G) \leq k - 1$. Consider vertices u, v in different connected components of $G - S$, then every u - v path in G uses an edge from S and hence there are at most $|S| \leq k - 1$ many edge-disjoint u - v paths in G .

(\Rightarrow) If $\lambda(G) = 0$, then the claim holds trivially, so we may suppose that $\lambda(G) \geq 1$. For $v \in V(G)$, define $E_v = \{e \in E(G) : e \ni v\}$ to be the set of edges incident to v . Since $|E_v| = \deg v \geq \delta(G) \geq \lambda(G) \geq 1$, we know that each E_v is non-empty.

Fix any $u \neq v \in V(G)$; we must find at least $\lambda(G)$ many edge-disjoint u - v paths in G . Consider the line graph $L(G)$ of G and consider the sets E_u and E_v (which are subsets of the vertices of $L(G)$). We claim that $\kappa_{L(G)}(E_u, E_v) \geq \lambda(G)$. Take any $S \subseteq E(G) = V(L(G))$ which is an E_u - E_v separator in $L(G)$. If $L(G) - S$ is connected, then either $S \supseteq E_u$ or $S \supseteq E_v$ which would mean that $|S| \geq \delta(G) \geq \lambda(G)$ as needed. Otherwise, $L(G) - S$ is disconnected. Now, $L(G - S) = L(G) - S$ and so DS1.7.4 implies that $G - S$ must be disconnected as well. Hence S is an edge-cut of G and so $|S| \geq \lambda(G)$ as needed.

We therefore know that $\kappa_{L(G)}(E_u, E_v) \geq \lambda(G)$ and so $p_{L(G)}(E_u, E_v) \geq \lambda(G)$ as well thanks to Lemma 1. Thus, there are at least $\lambda(G)$ vertex-disjoint E_u - E_v paths in $L(G)$. These paths correspond to disjoint sets of edges in G . While these sets of edges don't necessarily form u - v paths, each one contains a u - v path thanks to Lemma 4. Thus, we have found at least $\lambda(G)$ edge-disjoint u - v paths in G . \square