Here are two different (but very similar) proofs that a connected graph has a spanning tree. The ideas here can be very useful.

Theorem 1. *If G is connected, then G contains a spanning tree.*

Proof #1. Let $\mathcal{G}$ denote the set of all spanning subgraphs of G which are connected. Observe that $\mathcal{G}$ is non-empty since certainly $G \in \mathcal{G}$. Therefore, let $T \in \mathcal{G}$ be any element with the fewest number of edges. We claim that T is a spanning tree of G.

Firstly, since $T \in \mathcal{G}$, we know that T is a spanning subgraph of G and is connected; thus we need only show that T is acyclic. Suppose for the sake of contradiction that T contains a cycle; call it C. Fix any edge $e \in E(C)$ and consider $T' = T - e$. Since we did not modify the vertex set by removing the edge e, we know that T' is still a spanning subgraph of G. Additionally, T' is still connected since e was chosen to be in a cycle of T. But this means that $T' \in \mathcal{G}$; a contradiction since T' has strictly fewer edges than does T. □

Proof #2. Let $\mathcal{G}$ denote the set of all spanning subgraphs of G which are acyclic. Observe that $\mathcal{G}$ is non-empty since $(V(G), \varnothing) \in \mathcal{G}$. Therefore, let $T \in \mathcal{G}$ be any element with the maximum number of edges. We claim that T is a spanning tree of G.

Firstly, since $T \in \mathcal{G}$, we know that T is a spanning subgraph of G and is acyclic; thus we need only show that T is connected. Suppose for the sake of contradiction that T is disconnected; thus we can partition $V(T) = A \sqcup B$ with both A and B non-empty such that there are no edges of T between A and B. Since $V(T) = V(G)$ and G is connected, there must be some edge $e \in E(G) \setminus E(T)$ such that e has one end-point in A and the other in B; consider $T' = T + e$. Firstly, T' is still a spanning subgraph of G since we did not modify the vertex set. Next, T' is still acyclic since the edge e must have had its end points in different connected components of T. But this means that $T' \in \mathcal{G}$; a contradiction since T' has strictly more edges than does T. □

Corollary 2. *If G is a connected graph on n vertices, then $|E(G)| \geq n - 1$ with equality if and only if G is a tree.*

Proof. Theorem 1 guarantees that G has a spanning tree, call it T. Since T is a tree on n vertices, we know that $|E(T)| = n - 1$. Therefore, $n - 1 \leq |E(T)| \leq |E(G)|$ with equality if and only if $G = T$. □

Let's see another application of these ideas. The following theorem says that, in a connected graph, any set of edges can be extended to a spanning subgraph without introducing any extra cycles. The most common application of the theorem is that any acyclic set of edges can be extended to a spanning tree.

Theorem 3. *Let G be a connected graph and let S be any subset of edges of G. Then G has a connected, spanning subgraph H such that $E(H) \supseteq S$ and if C is a cycle in H, then $E(C) \subseteq S$.*

Before we prove the theorem, notice that Theorem 1 follows as a corollary by taking $S = \varnothing$.

Proof. Let $\mathcal{G}$ denote the set of all H such that

- H is a spanning subgraph of G, and

- H is connected, and
- $E(H) \supseteq S$.

We note that $\mathcal{G}$ is non-empty since $G \in \mathcal{G}$. Thus, let $H \in \mathcal{G}$ be any element with the fewest number of edges. We claim that H is our desired subgraph.

Firstly, since $H \in \mathcal{G}$, we know that H is a spanning subgraph of G, is connected and $E(H) \supseteq S$. So we need only show that H does not have any extraneous cycles. Suppose that C is a cycle in H with $E(C) \not\subseteq S$. Therefore, there is some edge $e \in E(C) \setminus S$; consider $H' = H - e$. Since we did not modify the vertex set by removing e, H' is a spanning subgraph of G. Additionally, $E(H') \supseteq S$ since $e \notin S$. Finally, H' is still connected since e was chosen to be in a cycle of H. But this means that $H' \in \mathcal{G}$; a contradiction since H' has strictly fewer edges than does H. $\square$