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The proof of Theorem 4.2 in the book skipped important details, and these details are not trivial, so here is a full proof.

**Theorem 1.** *Let  $G$  be a graph. If there are  $x, y \in V(G)$  for which there are at least two  $x$ - $y$  paths in  $G$ , then  $G$  contains a cycle.*

We give two proofs.

*Proof #1.* Notice that there is always exactly one  $x$ - $x$  path for any vertex  $x$ , and so we must have  $x \neq y$ . Call the two of the  $x$ - $y$  paths  $(x = u_0, u_1, \dots, u_k = y)$  and  $(x = v_0, v_1, \dots, v_\ell = y)$ ; note that  $k, \ell \geq 1$  since  $x \neq y$ .

Let  $i$  be the largest index for which  $u_j = v_j$  for all  $j \in \{0, \dots, i\}$ . Note that  $i$  exists since  $u_0 = x = v_0$ . Additionally, we see that  $i < \min\{k, \ell\}$ . Indeed, suppose, without loss of generality, that  $k \leq \ell$  and that  $i = k$ . If  $k = \ell$ , then this means that  $u_j = v_j$  for all  $j$  and so the two paths are the same, which we know is not the case. If  $k < \ell$ , then  $y = u_k = v_k$ , but this is impossible since  $v_k \neq v_\ell = y$ .

Therefore,  $u_{i+1}$  and  $v_{i+1}$  exist and  $u_{i+1} \neq v_{i+1}$ . Now, let  $s \in \{i+1, \dots, k\}$  be the smallest index for which  $u_s \in \{v_{i+1}, \dots, v_\ell\}$ . Note that  $s$  exists since  $u_k = y = v_\ell$ . Thus, suppose that  $t \in \{i+1, \dots, \ell\}$  is such that  $u_s = v_t$ . We know that either  $s \neq i+1$  or  $t \neq i+1$  since  $u_{i+1} \neq v_{i+1}$  from above.

We therefore see that  $(v_i = u_i, u_{i+1}, \dots, u_s = v_t, v_{t-1}, \dots, v_{i+1})$  forms a cycle in  $G$  as needed.  $\square$

*Proof #2.* Among all pairs of vertices with at least two paths between them, let  $x$  and  $y$  be a pair with  $d(x, y)$  minimum. Of course,  $d(x, y) \geq 1$  since there is exactly one path from a vertex to itself. Set  $d = d(x, y)$  and let  $(x = u_0, u_1, \dots, u_d = y)$  be any  $x$ - $y$  geodesic. Since there are at least two  $x$ - $y$  paths, we can find a different path, call it  $(x = v_0, v_1, \dots, v_k = y)$ . We claim that  $\{u_1, \dots, u_{d-1}\} \cap \{v_1, \dots, v_{k-1}\} = \emptyset$ . Suppose for the sake of contradiction that these sets intersect, and so  $u_s = v_t$  for some  $s \in [d-1]$  and  $t \in [k-1]$ . Since  $(x = u_0, \dots, u_d = y)$  is a geodesic, note that  $d(x, u_s) = s$  and  $d(u_s, y) = d - s$ . Now,  $W_1 = (x = u_0, \dots, u_s)$  and  $W_2 = (x = v_0, \dots, v_t = u_s)$  are two  $x$ - $u_s$  paths and  $W_3 = (u_s, u_{s+1}, \dots, u_d = y)$  and  $W_4 = (u_s = v_t, v_{t+1}, \dots, v_k = y)$  are two  $u_s$ - $y$  paths. While it could be the case that  $W_1 = W_2$  or that  $W_3 = W_4$ , it cannot be the case that both of these equalities hold since the original paths were distinct.

Case 1:  $W_1 \neq W_2$ . Then  $W_1$  and  $W_2$  are two different  $x$ - $u_s$  paths; a contradiction to the minimality of  $x, y$  since  $d(x, u_s) = s < d$ .

Case 2:  $W_3 \neq W_4$ . Then  $W_3$  and  $W_4$  are two different  $u_s$ - $y$  paths; a contradiction to the minimality of  $x, y$  since  $d(u_s, y) = d - s < d$ .

Therefore,  $\{u_1, \dots, u_{d-1}\} \cap \{v_1, \dots, v_{k-1}\} = \emptyset$  and so  $(x = u_0, u_1, \dots, u_d = y = v_k, v_{k-1}, \dots, v_1)$  forms a cycle in  $G$ .  $\square$