This homework is from https://mathematicaster.org/teaching/graphs2022/hw9.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (1 + 2 pts)**.** Fix any integer $n \geq 3$.

1. Construct an n-vertex graph G with $\binom{n-1}{2} + 1$ many edges such that G is *not* Hamiltonian.
   (Note: You must construct such a graph for *every* $n \geq 3$.)
2. Prove that if G is an n-vertex graph with at least $\binom{n-1}{2} + 2$ many edges, then G is Hamiltonian.

Problem 2 (0.5 + 0.5 pts)**.** Recall that K_{n_1,n_2,n_3} is the complete tripartite graph with parts of sizes n_1, n_2, n_3.

Fix any positive integer n.

1. Prove that $K_{n,2n,3n}$ is Hamiltonian.
2. Prove that $K_{n,2n,3n+1}$ is not Hamiltonian.

Problem 3 (2pts)**.** Let G be a graph on $n \geq 4$ vertices with the property that $N(u) \cup N(v) \supseteq V(G) \setminus \{u, v\}$ for every $u \neq v \in V(G)$. Prove that G is Hamiltonian.

(Hint: $|A \cup B| = |A| + |B| - |A \cap B|$ for finite sets A, B.)

(Hint: The problem and first hint suggest attempting a proof similar to our proof of Bondy–Chvátal, i.e. extending a Hamiltonian path to a Hamiltonian cycle. While such a proof is possible, it's more difficult than a more direct proof using only Ore's condition. This is just my opinion and maybe you disagree; I just don't want to lead you down the wrong path.)

Problem 4 (2pts)**.** Let G be a graph on n vertices. In class, we showed that $\alpha'(G) + \beta'(G) = n$ provided G has no isolated vertices; this exercise exists to establish the natural (and easier to prove) vertex-version of this fact.

Recall that the independence number of G, denoted by $\alpha(G)$, is the size of a largest independent set of G. Recall also that the vertex-cover number of G, denoted by $\beta(G)$, is the size of a smallest vertex-cover of G.

Prove that $\alpha(G) + \beta(G) = n$ for any n-vertex graph G.

(Note: You don't need to know anything about matchings to prove this.)

Problem 5 (2 pts)**.** We have an $m \times n$ matrix $M \in \{0, 1\}^{m \times n}$. We refer to the columns and rows of this matrix as *lines* (so a line is either one of the m rows or one of the n columns of M). Prove that the minimum number of lines needed to contain all 1's of M is precisely the maximum number of 1's in M that one can select so that no two of these selected 1's live in any common line.