This homework is from https://mathematicaster.org/teaching/graphs2022/hw8.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

We say that a graph is *even-regular* if every vertex has even degree, and we say that a graph is *odd-regular* if every vertex has odd degree.

Problem 1 (2pts)**.** In the second week of class (01-27), we used the handshaking lemma to prove the following fact: If G is a connected even-regular graph, then G has no bridges.

Give an alternative proof of this fact using what we now know about Eulerian circuits.

Problem 2 (2 + 2 pts)**.** A digraph D is said to be *oriented* if $(u,v) \in E(D) \implies (v,u) \notin E(D)$, i.e. D has no directed cycles of length 1 or 2.

For a digraph D with no loops, the *underlying simple graph* of D is the (simple) graph G with $V(G) = V(D)$ and $uv \in E(G)$ if and only if $(u,v) \in E(D)$ or $(v,u) \in E(D)$. In other words, the underlying simple graph is formed by simply forgetting about the directions of the edges.

For a (simple) graph G, an *orientation* of G is an oriented digraph whose underlying simple graph is G. In other words, an orientation of G is formed by assigning a direction to each edge of G. Note that a graph generally has many orientations.

1. Prove that if G is a (simple) even-regular graph, then G has an orientation wherein $\deg^+ v = \deg^- v$ for all vertices v.

2. Prove that if G is any (simple) graph, then G has an orientation wherein $|\deg^+ v - \deg^- v| \leq 1$ for all vertices v.

   You are free use part 1 as a black-box even if you haven't proved it.

Problem 3 (2 + 2 pts)**.** For graphs G, H, the *Cartesian product* of G and H is the graph $G \,\square\, H$ which has vertex set $V(G) \times V(H)$ and $\{(u_1, v_1), (u_2, v_2)\} \in E(G \,\square\, H)$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$.[1]

Suppose that G and H are any graphs.

1. Prove that $G \,\square\, H$ is connected if and only if both G and H are connected.

2. Prove that $G \,\square\, H$ is Eulerian if and only if both G and H are connected and also:

   (a) Both G and H are even-regular, or

   (b) Both G and H are odd-regular.

   You are free to use part 1 as a black-box even if you haven't proved it.

[1] N.b. We like the notation $\square$ here since $K_2 \,\square\, K_2 \cong C_4$, which looks like a $\square$. Note that your book uses $\times$ in place of $\square$; this is okay, but not desirable since generally $\times$ denotes a different graph product known as the categorical product, in which $K_2 \times K_2 \cong K_2 \sqcup K_2$, which can be made to look like an $\times$.