This homework is from https://mathematicaster.org/teaching/graphs2022/hw7.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** For infinitely many integers n, construct a graph G with the following properties:

- G is connected and has n vertices, and
- G is *not* a tree, and
- Every $v \in V(G)$ which is not a leaf of G is a cut-vertex of G.

"For infinitely many integers n..." means the following: Pick your favorite infinite subset $A \subseteq \mathbb{N}$ and, for each $n \in A$, build the desired graph for that n. For instance, maybe you pick A to be the set of even naturals, or maybe you pick A to be the set of integers of the form 2^k, or maybe you pick A to be the set of all primes larger than $100^{100^{100^{100}}}$, etc. As long as A is infinite, you're fine.

Problem 2 (1 + 1 pts)**.** Let G be a graph and suppose that H is any spanning subgraph of G.

1. Prove that $\lambda(G) \geq \lambda(H)$.
2. Prove that $\kappa(G) \geq \kappa(H)$.

Problem 3 (1 + 2 pts)**.** Let $G = (V, E)$ be a graph with at least one edge and fix any $e \in E$.

1. Prove that $\lambda(G) \geq \lambda(G - e) \geq \lambda(G) - 1$.
2. Prove that $\kappa(G) \geq \kappa(G - e) \geq \kappa(G) - 1$.

Problem 4 (1 pts)**.** For each non-negative integer k, find an example of a graph G with $\kappa(G) = \lambda(G) = 1$, yet there is some vertex $v \in V(G)$ such that $\kappa(G - v) = \lambda(G - v) = k$.

That is to say, the natural analogue of Problem 3 fails when deleting vertices instead of edges.

Problem 5 (2 pts)**.** Let G be a graph. The *kth power of* G is the graph G^k which has the same vertex-set as G and $uv \in E(G^k)$ iff $d_G(u, v) \leq k$.

Prove that if G is a connected graph on at least $k + 1$ vertices, then G^k is k-connected.

(You don't have to turn this is, but you should convince yourself that it's true: G^k is a clique if and only if $\text{diam}(G) \leq k$.)