This homework is from https://mathematicaster.org/teaching/graphs2022/hw5.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (0.5 + 0.5 + 1 pts)**.** For a fixed integer $k \geq 3$, a graph G is said to have property $\mathcal{C}_k$ if every subgraph of G with at least k edges contains a cycle. For a fixed integer $k \geq 4$, a graph G is said to have property $\mathcal{E}_k$ if every subgraph of G with at least k edges contains an even cycle.

1. For each $k \geq 3$, construct a graph G which has property $\mathcal{C}_k$ and has $|E(G)| = \binom{k}{2}$.
2. For each $k \geq 4$, construct a graph G which has property $\mathcal{E}_k$ and has $|E(G)| = \lceil k/2 \rceil \lfloor k/2 \rfloor$.
3. Prove that if G is a connected graph with property $\mathcal{C}_k$, then $|E(G)| \leq \binom{k}{2}$.

Problem 2 (3 pts)**.** This problem expands on the observation that trees on at least two vertices have at least two leaves.

Let T be a tree on at least two vertices. Let $\ell(T)$ denote the number of leaves of T and define $D_{\geq 2} = \{v \in V(T) : \deg v \geq 2\}$. Prove that

$$\ell(T) = 2 + \sum_{v \in D_{\geq 2}} (\deg v - 2)$$

Problem 3 (3 pts)**.** Fix an integer $n \geq 2$. Prove that a sequence of integers $d_1, \ldots, d_n$ is the degree sequence of some tree if and only if $d_i \geq 1$ for all $i \in [n]$ and $\sum_{i=1}^{n} d_i = 2n - 2$.

Problem 4 (2 pts)**.** Let G be a connected graph and let $w \colon E(G) \to \mathbb{R}$ be a weight function. Show that if all weights are distinct (that is $w(e) \neq w(s)$ for all distinct $e, s \in E(G)$), then G has a *unique* minimum spanning tree.