

This homework is from <https://mathematicaster.org/teaching/graphs2022/hw4.pdf>

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts). Let G and H be graphs. A *graph homomorphism* from G to H is a function $f: V(G) \rightarrow V(H)$ such that $\{f(u), f(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$.¹

Prove that a graph G is bipartite if and only if there is a graph homomorphism from G to K_2 .

Problem 2 (2 pts). A graph G is called *self-complementary* if $G \cong \overline{G}$. For example, P_4 and C_5 are self-complementary.

Prove that if G is a self-complementary graph on n vertices, then n is congruent to either 0 or 1 modulo 4.

Problem 3 (2 pts). Prove that a graph G on n vertices with $\delta(G) \geq 3$ must contain a cycle of length at most $\lfloor n/2 \rfloor + 1$.

Problem 4 (2 pts). Let T be a tree on n vertices with $\Delta(T) \leq 2$. Prove that $T \cong P_n$.

Problem 5 (2 pts). Determine (with proof) all trees T (up to isomorphism) on $n \geq 2$ vertices such that \overline{T} is also a tree. (Note: we do not require that $T \cong \overline{T}$.)

¹Note that a graph homomorphism does *not* need to be a bijection and that it could be the case that $\{f(u), f(v)\} \in E(H)$ even though $\{u, v\} \notin E(G)$.