This homework is from https://mathematicaster.org/teaching/graphs2022/hw12.pdf

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 + 2 + 2 pts)**.**

1. Let G be a bipartite graph with parts A, B where $|A| = |B| = n \geq 1$. Prove that if $|E(G)| > n(n-1)$, then G has a perfect matching.

   (It may be easier to rely on Kőnig here instead of on Hall, but it's up to you. You could even just induct on n.)

2. Let G be a bipartite graph with parts A, B and fix an integer $n \geq 1$. Let $A = A_1 \sqcup \cdots \sqcup A_n$ and $B = B_1 \sqcup \cdots \sqcup B_n$ be any partitions (some of the A_i's or B_j's may be empty). Note that $|A|$ and $|B|$ have nothing to do with n; n is just the number of pieces in each partition.

   Prove that if $|E(G)| < n$, then there is a bijection $\pi\colon [n] \to [n]$ such that G has no edges between A_i and $B_{\pi(i)}$ for each $i \in [n]$.

   You are free to use part 1 as a black-box even if you haven't proved it.

3. For each positive integer t, prove that if G is a t-critical graph, then $\lambda(G) \geq t - 1$.

   (Hint: Consult the notes from 03-01. Use part 2 to "merge independent sets".)

   You are free to use parts 1 and/or 2 as a black-box even if you haven't proved then.

   N.b. Vertex-connectivity is a very different story... In particular, the obvious analogue for vertex-connectivity is false (in general). The "Moser spindle" is a counter-example when $t = 4$, and there are many, many others.

Problem 2 (2pts)**.** Let $g \geq 2$ be an integer and let G be a connected plane graph on n vertices wherein every face is bounded by a cycle of G. Prove that if G has no cycles of length g or smaller, then

$$|E(G)| \leq \frac{g+1}{g-1}(n-2).$$

Problem 3 (2pts)**.** Prove a special case of the 4-color theorem: If G is a planar, triangle-free graph, then $\chi(G) \leq 4$.