

This homework is from <https://mathematicaster.org/teaching/graphs2022/hw1.pdf>

Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts). Let S be a finite set of integers and let G be the graph on vertex set S where for any $x, y \in S$, $xy \in E(G)$ if and only if $x + y$ is odd.

1. Prove that G is a bipartite graph for any such S .
2. If $S = [100]$, what is $|E(G)|$? Justify your answer.

Problem 2 (2 pts). Suppose that $(u = v_0, v_1, \dots, v_k = v)$ is a u - v geodesic. Prove that $d(u, v_i) = i$ for all $i \in \{0, \dots, k\}$.

Problem 3 (2 pts). Let G be a graph. For two non-empty subsets $A, B \subseteq V(G)$, an A - B path is a path in G which connects some vertex of A to some vertex of B . Prove that if P is a minimal A - B path, then P contains exactly one vertex from A and contains exactly one vertex from B .

Problem 4 (2 pts). Let G be a connected graph. Prove that any two maximum paths in G must share some vertex.

Problem 5 (2 pts). Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$. (recall that an independent set is a set of vertices which induce no edges)