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Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

We say that a graph is *even-regular* if every vertex has even degree, and we say that a graph is *odd-regular* if every vertex has odd degree.

Problem 1 (2pts). In the second week of class (01-27), we used the handshaking lemma to prove the following fact: If G is a connected even-regular graph, then G has no bridges.

Give an alternative proof of this fact using what we now know about Eulerian circuits.

Problem 2 (2 + 2 pts). A digraph D is said to be *oriented* if $(u, v) \in E(D) \implies (v, u) \notin E(D)$, i.e. D has no directed cycles of length 1 or 2.

For a digraph D with no loops, the *underlying simple graph* of D is the (simple) graph G with $V(G) = V(D)$ and $uv \in E(G)$ if and only if $(u, v) \in E(D)$ or $(v, u) \in E(D)$. In other words, the underlying simple graph is formed by simply forgetting about the directions of the edges.

For a (simple) graph G , an *orientation* of G is an oriented digraph whose underlying simple graph is G . In other words, an orientation of G is formed by assigning a direction to each edge of G . Note that a graph generally has many orientations.

1. Prove that if G is a (simple) even-regular graph, then G has an orientation wherein $\deg^+ v = \deg^- v$ for all vertices v .
2. Prove that if G is any (simple) graph, then G has an orientation wherein $|\deg^+ v - \deg^- v| \leq 1$ for all vertices v .

You are free use part 1 as a black-box even if you haven't proved it.

Problem 3 (2 + 2 pts). For graphs G, H , the *Cartesian product* of G and H is the graph $G \square H$ which has vertex set $V(G) \times V(H)$ and $\{(u_1, v_1), (u_2, v_2)\} \in E(G \square H)$ if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$.¹

Suppose that G and H are any graphs.

1. Prove that $G \square H$ is connected if and only if both G and H are connected.
2. Prove that $G \square H$ is Eulerian if and only if both G and H are connected and also:
 - (a) Both G and H are even-regular, or
 - (b) Both G and H are odd-regular.

You are free to use part 1 as a black-box even if you haven't proved it.

¹N.b. We like the notation \square here since $K_2 \square K_2 \cong C_4$, which looks like a \square . Note that your book uses \times in place of \square ; this is okay, but not desirable since generally \times denotes a different graph product known as the categorical product, in which $K_2 \times K_2 \cong K_2 \sqcup K_2$, which can be made to look like an \times .