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Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts). For infinitely many integers n , construct a graph G with the following properties:

- G is connected and has n vertices, and
- G is *not* a tree, and
- Every $v \in V(G)$ which is not a leaf of G is a cut-vertex of G .

“For infinitely many integers n ...” means the following: Pick your favorite infinite subset $A \subseteq \mathbb{N}$ and, for each $n \in A$, build the desired graph for that n . For instance, maybe you pick A to be the set of even naturals, or maybe you pick A to be the set of integers of the form 2^k , or maybe you pick A to be the set of all primes larger than $100^{100^{100}}$, etc. As long as A is infinite, you’re fine.

Problem 2 (1 + 1 pts). Let G be a graph and suppose that H is any spanning subgraph of G .

1. Prove that $\lambda(G) \geq \lambda(H)$.
2. Prove that $\kappa(G) \geq \kappa(H)$.

Problem 3 (1 + 2 pts). Let $G = (V, E)$ be a graph with at least one edge and fix any $e \in E$.

1. Prove that $\lambda(G) \geq \lambda(G - e) \geq \lambda(G) - 1$.
2. Prove that $\kappa(G) \geq \kappa(G - e) \geq \kappa(G) - 1$.

Problem 4 (1 pts). For each non-negative integer k , find an example of a graph G with $\kappa(G) = \lambda(G) = 1$, yet there is some vertex $v \in V(G)$ such that $\kappa(G - v) = \lambda(G - v) = k$.

That is to say, the natural analogue of Problem 3 fails when deleting vertices instead of edges.

Problem 5 (2 pts). Let G be a graph. The k th power of G is the graph G^k which has the same vertex-set as G and $uv \in E(G^k)$ iff $d_G(u, v) \leq k$.

Prove that if G is a connected graph on at least $k + 1$ vertices, then G^k is k -connected.

(You don’t have to turn this is, but you should convince yourself that it’s true: G^k is a clique if and only if $\text{diam}(G) \leq k$.)