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Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** Let G be any graph. Prove that

$$|E(G)| \geq \binom{\chi(G)}{2}.$$

Problem 2 (1pt)**.** Generalize Theorem 1 from 04-07:

Let $G = (V, E)$ be any graph and consider coloring the edges of G red and blue (formally, we have a function $f\colon E \to \{\text{red}, \text{blue}\}$). Let G_r be the graph formed by the red edges and let G_b be the graph formed by the blue edges (formally, $G_r = \big(V, f^{-1}(\text{red})\big)$ and $G_b = \big(V, f^{-1}(\text{blue})\big)$). Prove that

$$\chi(G_r) \cdot \chi(G_b) \geq \chi(G).$$

Problem 3 (1pt)**.** For every pair of positive integers m, n, construct a graph G with the following properties:

- G has $m \cdot n$ many vertices, and
- $\chi(G) = m$, and
- $\chi(\overline{G}) = n$.

Problem 4 (2pts)**.** Prove that G is 3-critical if and only if $G \cong C_{2n+1}$ for some positive integer n.

Problem 5 (2pts)**.** Fix any integer $k \geq 1$. Prove that if G is an n-vertex graph wherein

$$\chi\big(G[N(v)]\big) \leq k, \quad \text{for every } v \in V(G),$$

then $\chi(G) \leq \sqrt{2kn}$.

(Hint: Take motivation from our proof that $\chi(G) \leq \sqrt{2n}$ if G is triangle-free. Be warned, though, there are a couple steps which require more care.)

(Hint: If $0 \leq x \leq y$, then $x \leq \sqrt{xy}$.)

Problem 6 (2pts)**.** For graphs G, H, the graph G is said to be H-free if G does not contain a copy of H.

Fix any integer $k \geq 3$. Prove that if G is a C_k-free graph on n vertices, then $\chi(G) \leq \sqrt{2(k-2)n}$.

(Hint: Problem 5)