I don't want you to feel bored over Spring break, so here's a cute little problem you can spend some time with. It doesn't have much to do with anything we've been discussing so far, but it's still fun, especially if you enjoy matrices!

Firstly, for a graph G and vertices $u, v \in V(G)$, the *common neighborhood* of u and v is the set $N(u) \cap N(v)$.

Recall that the handshaking lemma implies that if G is a graph wherein every vertex has odd degree (i.e. the size of every neighborhood is odd), then G has an even number of vertices. Compare and contrast this with the following theorem:

Theorem 1. *If G is a graph wherein the common neighborhood of u and v has odd size for every $u \neq v \in V(G)$, then G has an odd number of vertices.*

I'll walk you through the steps of a proof.

Problem 1. Suppose that G satisfies the hypothesis of Theorem 1. Prove that $\deg v$ is even for all $v \in V(G)$.

Hint: Consider the induced subgraph $G' = G[N(v)]$ and consider $\deg_{G'} u$ for any $u \in N(v)$.

We now turn to a little bit of linear algebra. Throughout what follows, we assume that the vertex-set of G is $[n]$. The *adjacency matrix* of G is the matrix $A \in \{0,1\}^{n\times n}$ where $A_{ij} = 1$ if and only if $ij \in E(G)$. Note that $A_{ii} = 0$ for every $i \in [n]$ since no vertex is adjacent to itself.

Problem 2. For any $i, j \in [n]$ and any integer $k \geq 0$, show that $(A^k)_{ij}$ is precisely the number of i-j walks of length k in G.

Hint: Proceed by induction on k and explicitly work with the definition of matrix multiplication. This also appears one of the excursion sections in the book in case you hit a brick-wall.

Problem 3. For any $i, j \in [n]$, show that $(A^2)_{ij} = |N(i) \cap N(j)|$. In particular, $(A^2)_{ii} = \deg i$ for all $i \in [n]$.

This isn't really related to this problem, but it's a fun fact: $\mathrm{tr}(A^2) = 2|E(G)|$ and $\mathrm{tr}(A^3) = 6 \cdot \#(\text{triangles in } G)$. Unfortunately, there isn't a good formula for $\mathrm{tr}(A^k)$ in terms of edges and cycles for larger values of k.

Problem 4. Suppose that every vertex of G has even degree. Show that if $x \in (2\mathbb{Z}+1)^n$, then $Ax \in (2\mathbb{Z})^n$.

Here, $2\mathbb{Z}$ is the set of all even integers and $2\mathbb{Z}+1$ is the set of all odd integers (reasonable notation, don't you think?).

Problem 5. Suppose that G satisfies the hypothesis of Theorem 1. Use Problems 1 and 3 to show that if n is even and $x \in (2\mathbb{Z}+1)^n$, then also $A^2x \in (2\mathbb{Z}+1)^n$.

Problem 6. Use Problems 4 and 5 to prove Theorem 1.

Hint: Fix any $x \in (2\mathbb{Z}+1)^n$ (e.g. the all-ones vector) and compute A^2x in two different ways to reach a contradiction if n is even.