I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. For each odd integer $n \geq 1$, construct a graph G on n vertices for which both $\chi(G)$ and $\chi(\overline{G})$ are at least $(n+1)/2$.

This shows that the Nordhaus–Gaddum inequalities (Problem 2) are tight.

Problem 2 (Nordhaus–Gaddum inequalities)**.** Let G be a graph on n vertices. Prove that

$$\chi(G) \cdot \chi(\overline{G}) \leq \frac{(n+1)^2}{4}, \quad \text{and}$$
$$\chi(G) + \chi(\overline{G}) \leq n+1.$$

(Technically, the second inequality implies the first, but I think it's worth stating both of them)

The key idea behind both inequalities is to relate the degeneracy of G to that of $\overline{G}$. In particular, prove that $d(\overline{G}) \leq n - d(G) - 1$ and then use the fact that $\chi(H) \leq d(H) + 1$ to derive the stated inequalities.

Road map for showing that $d(\overline{G}) \leq n - d(G) - 1$:

1. Let H be a subgraph of G with $\delta(H) = d(G)$ and let H' be a subgraph of $\overline{G}$ with $\delta(H') = d(\overline{G})$.
2. Suppose for the sake of contradiction that $d(\overline{G}) \geq n - d(G)$ and argue that $V(H) \cap V(H') = \varnothing$.
3. Reach a contradiction by comparing $|V(H)|$ and $|V(H')|$.

Problem 3. Let D be a digraph with no loops. We define proper vertex-colorings of a digraph to be the same as proper vertex-colorings of its underlying simple graph (so we just forget about directions). In particular, $\chi(D)$ is the same as $\chi(G)$ where G is the underlying simple graph of D.

Let $p(D)$ denote the number of *vertices* in a longest directed path in D (recall that (x) is always a dipath which has 1 vertex).

1. Suppose that D is acyclic (has no directed cycles, though the underlying simple graph could have cycles). Prove that $\chi(D) \leq p(D)$.

   Hint: Let $f(v)$ denote the number of vertices in a longest dipath which ends at v. Show that f is a proper $p(D)$-coloring of D.

2. Show that part 1 still holds even if D contains dicycles.

   Hint: Take a maximally acyclic subgraph of D and apply the hinted f to this subgraph. Then show that every edge which was deleted when reducing to this subgraph is also properly colored under f.

3. A *tournament* of order n is simply an orientation of K_n. Show that every tournament contains a Hamiltonian dipath (a dipath which contains all vertices).

4. Let T be a tournament of order n and consider coloring the edges of T red and blue. Prove that T contains a monochromatic (all edges the same color) dipath on at least $\sqrt{n}$ many vertices.

Problem 4. Let G be a connected plane graph and suppose that every face of G has length either 5 or 6. If G is additionally 3-regular, show that G must have exactly 12 faces of length 5.

So it is no accident that soccer balls have exactly 12 pentagons on their surface!

Problem 5. Let G be a connected plane graph wherein every face is bounded by a cycle. Prove that if G has no cycles of length 5 or shorter, then $\chi(G) \leq 3$.

Is there any bound on the cycle lengths (e.g. forbidding all cycles of length less than $10^{10^{10^{10}}}$) that would imply that $\chi(G) \leq 2$?

Problem 6. Is there a graph G on exactly 6 vertices which is non-planar, yet does not contain a copy of K_5 nor $K_{3,3}$?

Problem 7. Let G be a graph. G contains vertices $v_1, \ldots, v_5$ where $\deg v_1 = 100$, $\deg v_2 = 30$, $\deg v_3 = 30$, $\deg v_4 = 4$, $\deg v_5 = 3$ and all other vertices of G have degree either 1 or 2. Knowing nothing else about G, can you determine whether or not G is planar?

Problem 8. The *crossing number* of G, denoted by $\mathrm{cr}(G)$ is the minimum number of pairs of edges of G that must cross when attempting to draw G in the plane. In particular $\mathrm{cr}(G) = 0$ iff G is planar. Similarly $\mathrm{cr}(G) = 1$ iff G is non-planar and there is a drawing of G in which exactly two of the edges cross (since cr counts *pairs* of crossing edges).

1. Show that $\mathrm{cr}(K_5) = \mathrm{cr}(K_{3,3}) = 1$.

2. Suppose that G is a graph with $n \geq 3$ vertices and m edges. Prove that $\mathrm{cr}(G) \geq m - 3n + 6$.

   (To make life easier, feel free to assume that Theorem 9 from 04-14 holds even if G is disconnected (it does still hold provided $n \geq 3$; we just didn't prove it))