I encourage you to first read through all of these problems and then focus first on those with which you're less comfortable.

Problem 1. Prove that if $\Delta(G) \le 2$, then $\kappa(G) = \lambda(G)$.

Problem 2. Recall that $\kappa(G) \le \lambda(G) \le \delta(G)$ for any graph G. Convince yourself that these inequalities can be strict. Furthermore, show that for any integers κ, λ, δ with $1 \le \kappa < \lambda < \delta$, there is a graph G with $\kappa(G) = \kappa$, $\lambda(G) = \lambda$ and $\delta(G) = \delta$.

Problem 3. Let G be a graph and let $A, B \subseteq V(G)$ be non-empty subsets (that could intersect). An A-B path is a path $(v_0, \ldots, v_k)$ with $v_0 \in A$, $v_k \in B$ and none of $v_1, \ldots, v_{k-1}$ are in either A or B (we used these paths in our proof of Menger's theorem). Note that (x) is an A-B path if and only if $x \in A \cap B$.

1. Use Menger's theorem for vertex-connectivity to prove that if $|A|, |B| \ge \kappa(G)$, then there are at least $\kappa(G)$ many vertex-disjoint A-B paths in G.

   N.b. This fact also follows immediately from Lemma 1 in the extra notes from 03-03; one just needs to observe that $\kappa_G(A, B) \ge \kappa(G)$ in this case.

2. Use Menger's theorem for edge-connectivity to prove that if $|A|, |B| \ge \lambda(G)$, then there are at least $\lambda(G)$ many edge-disjoint A-B paths in G such that each vertex in $A \cup B$ belongs to at most one of these paths (i.e. no two start at nor end at the same point).

Problem 4. Suppose that a graph G has blocks $B_1, \ldots, B_k$ and cut-vertices $v_1, \ldots, v_\ell$. We build a new graph $\mathcal{B}$ with $V(\mathcal{B}) = \{b_1, \ldots, b_k, c_1, \ldots, c_\ell\}$ and $c_i b_j \in E(\mathcal{B})$ if and only if $v_i \in V(B_j)$. Note that $\mathcal{B}$ is a bipartite graph with parts $\{b_1, \ldots, b_k\}$ and $\{c_1, \ldots, c_\ell\}$.

Prove that if G is a connected graph on at least two vertices, then $\mathcal{B}$ is a tree. Furthermore, show that none of $c_1, \ldots, c_\ell$ are leaves of $\mathcal{B}$. $\mathcal{B}$ is sometimes called the *block tree* of G.

Problem 5. The *degeneracy* of a graph G is defined to be

$$d(G) = \max\big\{\delta(H) : H \text{ is a subgraph of } G\big\}.$$

1. Prove that $d(G) \le 1$ if and only if G is a forest.

2. Prove that $d(G)$ is the smallest integer d such that there is an ordering $V(G) = \{v_1, \ldots, v_n\}$ so that $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \le d$ for all $i \in [n]$.

   Such orderings will be our friend once we start talking about coloring graphs :)

3. Prove that if G is connected and $d(G) = \Delta(G)$, then G is regular.

The *arboricity* of a graph G, denoted by $a(G)$, is defined to be the minimum integer k such that we can partition the edges of G into k forests. If G has no edges, we set $a(G) = 0$.

4. Prove that $d(G) \ge a(G)$.

5. Prove that $d(G) \le 2a(G) - 1$.

Problem 6. Let G be a graph and let $\mathcal{I}$ be any collection of independent sets of G. For a vertex $v \in V(G)$, let $\mathcal{I}_v = \{I \in \mathcal{I} : v \in I\}$ be the set of those independent sets in $\mathcal{I}$ which contain the vertex v. Say that $v \in V(G)$ is *uncommon* if $|\mathcal{I}_v| \leq |\mathcal{I}|/2$ and otherwise say that v is *common*.

1. Prove that if G has at least one edge, then G has an uncommon vertex.
2. Prove that if G is *not* bipartite, then there is an edge $uv \in E(G)$ such that both u and v are uncommon.
3. Open question: Suppose that $\mathcal{I}$ is the set of all maxim**al** independent sets of G. If G has at least one edge, then there is some $uv \in E(G)$ such that both u and v are uncommon.

Problem 7. If you read the supplementary notes on Dyck paths, we proved that there are at most 4^n non-isomorphic trees on n vertices. This exercise will establish that there are at least α^n many non-isomorphic trees on n vertices for some $\alpha > 1$.

1. Prove that there are at least $n^{n-2}/n!$ many non-isomorphic trees on n vertices.
2. Use the inequality $1 - x \leq e^{-x}$ (which can be proved via elementary calculus if you care to do so) to prove that $n! \leq n^n/e^{(n-1)/2}$.
3. Conclude that there are at least $e^{(n-1)/2}/n^2$ many non-isomorphic trees on n vertices, which is approximately 1.6487^n for large n.

N.b. With a more careful upper-bound on $n!$ which can be found by approximating $\log n!$ by an integral (see Stirling's approximation), one can improve the lower-bound to approximately $e^n \approx 2.7182^n$. As mentioned in the supplementary notes, the actual answer is approximately 2.9557^n.

Problem 8. Fix an integer $n \geq 2$ and let $d_1, \ldots, d_n$ be a sequence of positive integers with $\sum_{i=1}^n d_i = 2n - 2$. Prove that the number of (labeled) trees T with vertex-set $[n]$ and $\deg_T i = d_i$ for all $i \in [n]$ is precisely

$$\binom{n-2}{d_1 - 1, \ldots, d_n - 1}.$$ [1]

Problem 9. Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Consider the following "reverse-Kruskal algorithm":

Initialize $H = G$ and iterate the following process:

1. If H is acyclic, terminate and return H.
2. If H has a cycle, do the following. Let $\mathcal{C} \subseteq E(H)$ denote the set of all edges of H contained within a cycle of H. Take any edge $e \in \mathcal{C}$ of maximum weight, replace H by $H - e$ and repeat.

Prove that "reverse-Kruskal" returns a minimum spanning tree of G.

[1] For any non-negative integers $k_1, \ldots, k_\ell$ with $\sum_{i=1}^\ell k_i = r$,

$$\binom{r}{k_1, \ldots, k_\ell} = \left|\left\{(A_1, \ldots, A_\ell) \in \left(2^{[r]}\right)^\ell : [r] = \bigsqcup_{i=1}^\ell A_i \text{ and } |A_i| = k_i\right\}\right| = \frac{r!}{k_1! \cdots k_\ell!}.$$

If you haven't seen multinomial coefficients before, convince yourself that $\binom{n}{k, n-k} = \binom{n}{k}$ as a warm-up.

Problem 10. Let $T_1 \neq T_2$ be two trees on the same vertex set. For any edge $e \in E(T_1) \setminus E(T_2)$, we know that $T_2 + e$ contains a unique cycle; call this cycle C_e. Prove that

$$E(T_2) \setminus E(T_1) \subseteq \bigcup_{e \in E(T_1) \setminus E(T_2)} E(C_e).$$

Problem 11. Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Let $\mathcal{T}$ denote the set of all spanning trees of G and let $\mathcal{T}_{\min}$ denote the set of all minimum spanning trees of G.

1. Fix any $T_1 \in \mathcal{T}_{\min}$ and any $T_2 \in \mathcal{T}$ with $T_1 \neq T_2$. Prove that there is some $e \in E(T_2) \setminus E(T_1)$ and some $s \in E(T_1) \setminus E(T_2)$ such $T_3 = T_2 - e + s$ is a spanning tree of G and $w(T_3) \leq w(T_2)$.

2. Let $\mathcal{G}$ be the graph with vertex-set $\mathcal{T}$ where $T_1T_2 \in E(\mathcal{G})$ iff $|E(T_1) \triangle E(T_2)| = 2$.[2] Fix any $T \in \mathcal{T}$ and any $T' \in \mathcal{T}_{\min}$. Prove that there is a path $(T = T_0, \ldots, T_k = T')$ in $\mathcal{G}$ such that $w(T_i) \leq w(T_{i-1})$ for all $i \in [k]$.

3. Prove that $\mathcal{G}$ is a connected graph.

4. Let $\mathcal{G}_{\min}$ be the subgraph of $\mathcal{G}$ induced by $\mathcal{T}_{\min}$. Prove that $\mathcal{G}_{\min}$ is a connected graph.

Problem 12. Let G be a connected graph and let $w\colon E(G) \to \mathbb{R}$ be a weight function. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a strictly increasing function (that is, $x < y \iff f(x) < f(y)$) and consider a new weight function $w'\colon E(G) \to \mathbb{R}$ defined by $w' = f \circ w$. Prove that T is a minimum spanning tree with respect to w if and only if T is a minimum spanning tree with respect to w'.

Note: A naïve idea is to try to show that $x_1 + \cdots + x_k \leq y_1 + \cdots + y_k$ if and only if $f(x_1) + \cdots + f(x_k) \leq f(y_1) + \cdots + f(y_k)$. But this is false; indeed, $1 + 4 < 3 + 3$, yet $1^3 + 4^3 > 3^3 + 3^3$ (note that x^3 is a strictly increasing function). Instead use the key idea from Problem 11.

Problem 13. Let G be a connected graph. A vertex $c \in V(G)$ is called a *center* of G if $d(c, u) \leq \lceil \mathrm{diam}(G)/2 \rceil$ for all $u \in V(G)$ (understand why this definition is sensible).

1. Find an infinite collection of (non-isomorphic) connected graphs which have no center.

2. Prove that if T is a tree then T has a center. Furthermore, prove that:

   (a) If $\mathrm{diam}(T)$ is even, then T has a unique center.

   (b) If $\mathrm{diam}(T)$ is odd, then T has exactly two centers and these two centers form an edge of T.

3. (Please do part 1 before this part) A graph G is called *vertex-transitive* if for any pair of vertices $u, v \in V(G)$, there is some automorphism $f \in \mathrm{Aut}(G)$ with $f(u) = v$. Prove that if G is a connected, vertex-transitive graph which is not a clique, then G has no center.

   N.b. If you know a bit of group theory, then you can construct a diverse collection of vertex-transitive graphs known as Cayley graphs.

Problem 14. Let G be any disconnected, spanning subgraph of K_n and suppose that $G_1, \ldots, G_k$ are the connected components of G. Set $V_i = V(G_i)$; note that $k \geq 2$ since G is disconnected and that we could have $|V_i| = 1$ for some (or all) i's.

Let $\mathcal{S}$ denote the set of all (labeled) subgraphs H of K_n such that

[2] Recall that $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$; that is, $A \triangle B$ is the set of elements which are in either A or in B but not in both.

- H is a connected, spanning subgraph of K_n, and
- G is a subgraph of H, and
- If C is a cycle of H, then C is actually a cycle of G (that is, H contains no additional cycles).

Prove that

$$|\mathcal{S}| = n^{k-2} \prod_{i=1}^{k} |V_i|.$$

Hint #1: Letting $\mathcal{T}_k$ denote the set of all (labeled) trees on vertex-set $[k]$, consider the function $f\colon \mathcal{S} \to \mathcal{T}_k$ defined by, for $H \in \mathcal{S}$ and $i \neq j \in [k]$, $ij \in E(f(H))$ if and only if H has an edge with one vertex in G_i and the other in G_j. (You will need to show that f is well-defined, i.e. $f(H)$ is indeed a tree)

Hint #2: Problem 8 and the multinomial theorem will be helpful:

$$\left(\sum_{i=1}^{\ell} x_i\right)^r = \sum_{\substack{d_1,\ldots,d_\ell \in \mathbb{Z}_{\geq 0}: \\ d_1+\cdots+d_\ell = r}} \binom{r}{d_1, \ldots, d_\ell} \prod_{i=1}^{\ell} x_i^{d_i}.$$

Hint #2 (alternate): Alternatively, define a version of Prüfer codes that encapsulate this situation (hint #1 will still be helpful).