These notes are from https://mathematicaster.org/teaching/graphs2022/extra_04-28.pdf

Last time we showed that $R(n,n) \leq 4^n$. Today, we're going to work on proving lower bounds on $R(n,n)$. Remember that the strategies for proving upper- and lower-bounds on Ramsey numbers are very different. In order to prove that $R(n,n) \geq N$, we somehow need to construct a red,blue-coloring of $E(K_{N-1})$ which does not contain any monochromatic copy of K_n.

Theorem 1. *For every $n \geq 2$,*

$$R(n,n) \geq \sqrt{2}^n.$$

Compare this with our upper bound of $R(n,n) \leq 4^n$.

For the proof, recall the notion of indicator functions from a couple lectures ago: for a proposition P, the function $\mathbf{1}[P]$ is 1 if P is true and is 0 if P is false.

Proof. We first handle a couple small values of n:

- $R(2,2) = 2 \geq \sqrt{2}^2$.
- $R(3,3) = 6 \geq \sqrt{2}^3$.
- $R(4,4) = 18 \geq \sqrt{2}^4$.

Thus, for the remainder of the proof, suppose that $n \geq 5$.

Set $N = 2^{\lceil n/2 \rceil}$; we will show that there exists some red,blue-coloring of $E(K_N)$ which has no monochromatic copy of K_n. If we can accomplish this lofty goal, then we will have shown that $R(n,n) > N \geq \sqrt{2}^n$.

On quick note: the following proof is much cleaner if done using random variable and expectations. However, I don't expect that you've ever seen these. If/when you do come across these concepts, though, come back and re-write this proof!

At the current moment, there is no known way to actually build such a coloring in general (that is, explicitly describe what it looks like), so instead we will just show that it exists. We do so via averaging over all red,blue-colorings of $E(K_N)$. To this end, let $C(N)$ denote the set of all red,blue-colorings of $E(K_N)$. Note that $|C(N)| = 2^{\binom{N}{2}}$ each edge receives either color red or color blue.

Now, for any coloring $f \in C(N)$, observe that

$$\#\{\text{mono}\chi \text{ copies of } K_n \text{ in } f\} = \sum_{T \in \binom{[N]}{n}} \mathbf{1}\left[\binom{T}{2} \text{ is mono}\chi \text{ in } f\right].$$

In other words, we go to every copy of K_n in K_N and ask "are you monochromatic?" and add up all the "yes"s.

Now, we switch the order of summation to write

$$\begin{aligned}
\sum_{f\in C(N)} \#\{\text{mono}\chi \text{ copies of } K_n \text{ in } f\} &= \sum_{f\in C(N)} \sum_{T\in\binom{[N]}{n}} \mathbf{1}\left[\binom{T}{2} \text{ is mono}\chi \text{ in } f\right] \\
&= \sum_{T\in\binom{[N]}{n}} \sum_{f\in C(N)} \mathbf{1}\left[\binom{T}{2} \text{ is mono}\chi \text{ in } f\right] \\
&= \sum_{T\in\binom{[N]}{n}} \#\left\{\text{red,blue-colorings of } E(K_N) \text{ in which } \binom{T}{2} \text{ is mono}\chi\right\}.
\end{aligned}$$

Now for any fixed $T \in \binom{[N]}{n}$, we need to determine the number of ways to color $E(K_N)$ so that every edge in $\binom{T}{2}$ is the same color. First, there are two ways for $\binom{T}{2}$ to be monochromatic: either every edge is red or every edge is blue. Then, since $|T| = n$, we know that there are $\binom{N}{2} - \binom{n}{2}$ many edges not within T and so there are $2^{\binom{N}{2}-\binom{n}{2}}$ many ways to color these remaining edges. Thus, there are $2^{\binom{N}{2}-\binom{n}{2}+1}$ many red,blue-colorings of $E(K_N)$ in which $\binom{T}{2}$ is monochromatic. Continuing our equality train, we then have

$$\sum_{f\in C(N)} \#\{\text{mono}\chi \text{ copies of } K_n \text{ in } f\} = \sum_{T\in\binom{[N]}{n}} 2^{\binom{N}{2}-\binom{n}{2}+1} = \binom{N}{n} 2^{\binom{N}{2}-\binom{n}{2}+1}.$$

Now, ignore the $2^{\binom{N}{2}}$ piece for a moment and focus on $\binom{N}{n}2^{-\binom{n}{2}+1}$. We require two inequalities, which are pretty quick to verify:

- $\displaystyle\binom{N}{n} \le \frac{N^n}{n!}$.
- $2^{n+1} < n!$ whenever $n \ge 5$ (which is the case we're in).

Using these inequalities and the fact that $N = 2^{\lceil n/2\rceil}$, we bound

$$\begin{aligned}
\binom{N}{n} 2^{-\binom{n}{2}+1} &\le \frac{N^n}{n!} 2^{-\binom{n}{2}+1} = \frac{1}{n!} 2^{n\lceil n/2\rceil - n(n-1)/2+1} \\
&\le \frac{1}{n!} 2^{n(n+1)/2-n(n-1)/2+1} = \frac{1}{n!} 2^{n+1} < 1.
\end{aligned}$$

Now, $|C(N)| = 2^{\binom{N}{2}}$, so we have shown that

$$\begin{aligned}
\underset{f\in C(N)}{\text{average}}\ \#\{\text{mono}\chi \text{ copies of } K_n \text{ in } f\} &= 2^{-\binom{N}{2}} \sum_{f\in C(N)} \#\{\text{mono}\chi \text{ copies of } K_n \text{ in } f\} \\
&= 2^{-\binom{N}{2}} \cdot \binom{N}{n} 2^{\binom{N}{2}-\binom{n}{2}+1} = \binom{N}{n} 2^{-\binom{n}{2}+1} < 1.
\end{aligned}$$

Of course, not everyone can be above average, so there must be some red,blue-coloring of $E(K_N)$ with < 1 (i.e. 0) monochromatic copies of K_n! So whatever this coloring may be, it demonstrates that $R(n, n) > N = 2^{\lceil n/2\rceil} \ge \sqrt{2}^n$. $\square$

The moral of the above proof is: put all the colorings in a bag, grab one out at random and you'll almost certainly win! Indeed, we actually showed that the average number of monochromatic copies of K_n in a red,blue-coloring of $K_{\sqrt{2}^n}$ is at most $2^{n+1}/n!$ which tends toward 0 quite quickly as $n \to \infty$ ($2^{n+1}/n! \approx 1/n^n$), so the vast majority of such colorings have no monochromatic copy of K_n. Despite this, though, no one has any clue how to actually "build" such a coloring, that is, explicitly describe what one looks like. Basically, it's like trying to pick out a needle in a haystack made of needles, yet somehow only being able to find hay...

Corollary 2. *For each $n \geq 2$, there is a red,blue-coloring of $E(K_n)$ in which* every *monochromatic clique has at most $2\log_2 n$ many vertices.*

In graph land: There exists an n-vertex graph G wherein $\omega(G) \leq 2\log_2 n$ and $\alpha(G) \leq 2\log_2 n$.

Compare and contrast this with what we proved last time: *every* n-vertex graph G has either $\omega(G) > \frac{1}{2}\log_2 n$ or $\alpha(G) > \frac{1}{2}\log_2 n$.

Proof. Let k be the smallest integer for which $\sqrt{2}^{k+1} > n$ ($k \geq 2$ since $n \geq 2$). Then $R(k+1, k+1) \geq \sqrt{2}^{k+1} > n$, so we can find a red,blue-coloring of $E(K_n)$ which contains no monochromatic copy of K_{k+1}, i.e. every monochromatic clique has at most k vertices. Then, by the definition of k, we have

$$\sqrt{2}^k \leq n \implies k \leq 2\log_2 n.$$

□

Let's reiterate the bounds that we have on $R(n, n)$:

$$\sqrt{2}^n \leq R(n, n) \leq 4^n.$$

To this date, only very minor improvements on either of these bounds is known (and both bounds were derived in the 1930's). It would be utterly amazing if you could somehow show that either

$$R(n, n) \leq (4 - 0.00...01)^n, \quad \text{or} \quad R(n, n) \geq (\sqrt{2} + 0.00...01)^n,$$

for all $n \geq 100...00$. In addition to job security for life, I believe there are some monetary prizes available (not a million dollars, though).

Let's pivot and ask a slightly different question. Recall that $R(3, 3) = 6$. This means that if $n \geq 6$, then *any* red,blue-coloring of $E(K_n)$ contains a monochromatic triangle. But if n is large, then there should be much, much more than one such triangle. Pretty easily, there are at least $\lfloor n/6 \rfloor$ such monochromatic triangles since we could just break up K_n into 6-vertex chunks. This is pretty wasteful still; by optimizing this idea, you can find roughly $n/3$ monochromatic triangles. But even this is far from the actual answer!

Theorem 3. *Every red,blue-coloring of $E(K_n)$ contains at least*

$$\binom{n}{3} - \frac{n(n-1)^2}{8}$$

many monochromatic triangles.

Note that we don't require all of these triangles to be of the same color, we just care that the triangle itself is monochromatic.

Before we begin the proof, I think it's important to point out the following two consequences:

Note that $\binom{n}{3} = \frac{n(n-1)(n-2)}{6} \approx \frac{n^3}{6}$ when n is large, so

$$\binom{n}{3} - \frac{n(n-1)^2}{8} \approx \frac{n^3}{6} - \frac{n^3}{8} = \frac{n^3}{24} \approx \frac{1}{4}\binom{n}{3}.$$

Since K_n contains $\binom{n}{3}$ many triangles in all, this means that, when n is large, roughly a quarter of all triangles must be monochromatic in any red,blue-coloring of $E(K_n)$!

Also:

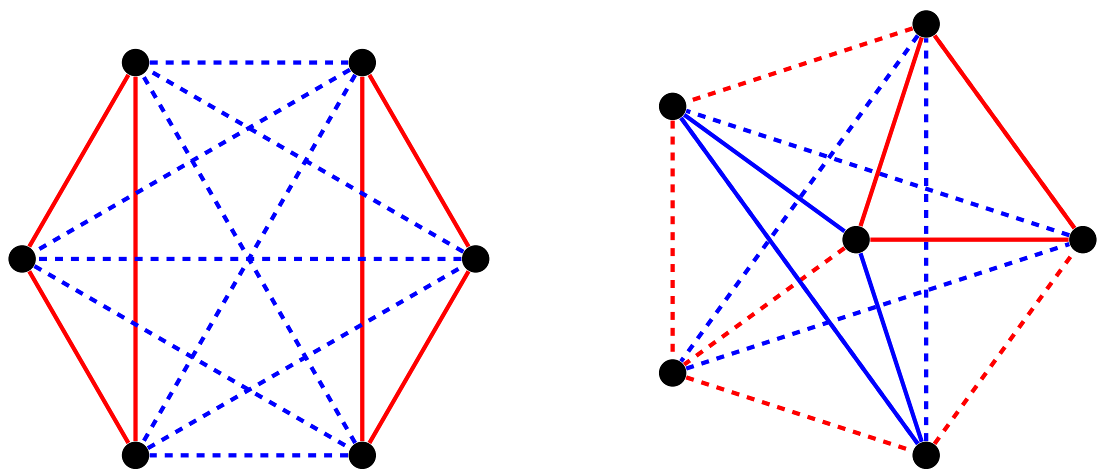
Corollary 4. *Any red,blue-coloring of $E(K_6)$ must contain at least 2 monochromatic triangles, which is optimal.*

Proof. Since $R(3,3) = 6$, we know that any red,blue-coloring of $E(K_6)$ must contain at least one monochromatic triangle. However, thanks to Theorem 3, there must be at least

$$\binom{6}{3} - \frac{6 \cdot 5^2}{8} = 20 - \frac{75}{4} = \frac{5}{4}$$

monochromatic triangles. Since the number of monochromatic triangles is an integer and $\frac{5}{4} > 1$, there must be at least two.

The optimality (i.e. there exist red,blue-colorings of $E(K_6)$ with exactly two monochromatic triangles) can be seen via either of the following two pictures (edges not in a monochromatic triangle are dashed for visibility):



$\square$

Proof of Theorem 3. Let f be any red,blue-coloring of $E(K_n)$. Note that there are $\binom{n}{3}$ many triangles in K_n over-all, so it is equivalent to show that the number of *non*-monochromatic triangles in f is at most $n(n-1)^2/8$.

Define the set $P \subseteq [n]^3$ by

$$P = \{(x, y, z) \in [n]^3 : f(xy) = \text{red and } f(xz) = \text{blue}\}.$$

In other words, P is essentially the set of incident red- and blue-edges.

Now, let $N \subseteq \binom{[n]}{3}$ denote the set of triples in K_n which induce *non*-monochromatic triangles; i.e. $xyz \in N$ if and only if there are at least two colors between the vertices x, y, z.

We begin by claiming that $|P| = 2|N|$. Indeed, for any $(x, y, z) \in P$, we find that xyz induces a non-monochromatic triangle, so $xyz \in P$. In the reverse, suppose that $xyz \in N$. Since xyz

induces a non-monochromatic triangle, we know that there are two edges of the same color and one edge of the opposite color (since there are three edges total). If, say, xy, yz are red and xz is blue, then $(x, y, z), (z, y, x) \in P$ and no other permutation of xyz resides within P. Every other case is essentially symmetric to this one, so we won't bother to verify them all (though you should check for yourself if you're unconvinced!).

Now that we know that $|P| = 2|N|$, we seek to understand $|P|$. For a vertex $v \in [n]$, let $\deg_r v$ and $\deg_b v$ denote the number of red- and blue-edges incident to v respectively. Note that $\deg_r v + \deg_b v = \deg v = n - 1$ since every edges is colored either red or blue.

To get an upper bound on $|P|$, we require the AM–GM inequality, which states that $xy \leq (x + y)^2/4$. For a quick proof:

$$\begin{aligned} &(x-y)^2 \geq 0 \\ \implies\ &x^2 - 2xy + y^2 \geq 0 \\ \implies\ &x^2 + 2xy + y^2 \geq 4xy \\ \implies\ &(x+y)^2 \geq 4xy. \end{aligned}$$

Now on to bounding $|P|$. By summing over the first coordinate and applying the AM–GM inequality, we find that

$$\begin{aligned} |P| &= \sum_{x \in [n]} \left|\{(y, z) \in [n]^2 : f(xy) = \text{red and } f(xz) = \text{blue}\}\right| \\ &= \sum_{x \in [n]} \deg_r x \cdot \deg_b x = \sum_{x \in [n]} \frac{(\deg_r x + \deg_b x)^2}{4} = \sum_{x \in [n]} \frac{(n-1)^2}{4} = \frac{n(n-1)^2}{4}. \end{aligned}$$

Thus, the number of non-monochromatic triangles in f is $|N| = \frac{1}{2}|P| \leq \frac{n(n-1)^2}{8}$, which yields the claim. $\square$

Recall that the above theorem implies that, for large n, any red,blue-coloring of $E(K_n)$ must have at least (approximately) $\frac{1}{4}\binom{n}{3}$ monochromatic triangles (i.e. at least roughly $1/4$ of all triangles are monochromatic). It turns out that this is (approximately) tight. By walking through an averaging argument similar to what we did at the start, one can find a red,blue-coloring with strictly fewer than $\frac{1}{4}\binom{n}{3}$ monochromatic triangles. I'll make this a worksheet question.