These notes are from https://mathematicaster.org/teaching/graphs2022/extra_04-26.pdf

Consider coloring the edges of K_n with red and blue, so we have a function $f\colon E(K_n) \to \{\text{red}, \text{blue}\}$. For a graph H, we say that the coloring f has a red copy of H if we can find a copy of H all of whose edges are colored red. Similarly, f has a blue copy of H if we can find a copy of H all of whose edges are colored blue. We also so that f has a monochromatic copy of H if it has either a red or a blue copy of H. All of these notions can be extended to more than two colors and to coloring the edges of graphs other than cliques. But let's stick with this situation for awhile.

We are interested in what monochromatic structures *must* appear in a red,blue-coloring of $E(K_n)$, no matter what that coloring actually is. For instance, if $n \geq 2$, then certainly there must always be a monochromatic edge, but we're interested in more interesting structures.

Definition 1. *For positive integers m, n, the* Ramsey number $R(m,n)$ *is the smallest integer N such that* every *red,blue-coloring of $E(K_N)$ contains either a red copy of K_m or a blue copy of K_n.*

If you don't like crayons, this is really equivalent to a pure graph property. Indeed, red,blue-colorings of $E(K_N)$ correspond to a graph and its complement on N vertices. To see this, simply let G be the "red-graph", then $\overline{G}$ is the "blue-graph". In the reverse direction, given any N-vertex graph G, create a red,blue-coloring of $E(K_N)$ by coloring the edge red if it lives in G and coloring the edge blue if it doesn't (i.e. it lives in $\overline{G}$).

Observation 2. *For positive integers m, n, $R(m,n)$ is the smallest integer N such that* every *N-vertex graph G has either $\omega(G) \geq m$ or $\alpha(G) \geq n$, i.e. G contains a copy of K_m or $\overline{G}$ contains a copy of K_n.*

A priori, it is not even clear that $R(m,n)$ even exists for all values of m, n! Why should it have to? We will soon prove that it does indeed exist — a fact known as Ramsey's theorem (well, really Ramsey's theorem is much, much more general).

Here are a few observations (assuming existence):

- If N is any integer with $N \geq R(m,n)$, then also every red,blue-coloring of $E(K_N)$ contains either a red copy of K_m or a blue copy of K_n since K_r is a subgraph of K_N whenever $r \leq N$.
- $R(m,n) = R(n,m)$ since we could swap red and blue.
- $R(1,n) = 1$ for every n since K_1 is only a single vertex.
- $R(2,n) = n$ for every n since *any* red edge would be a red copy of K_2 and we need at least n vertices to even have the chance at a blue copy of K_n (if we had only $n-1$ vertices, then we could color everything blue).
- If $m_1 \leq m_2$ and $n_1 \leq n_2$, then $R(m_1, n_1) \leq R(m_2, n_2)$.

Before we get into any proofs, I want to discuss the philosophy behind any argument about Ramsey numbers.

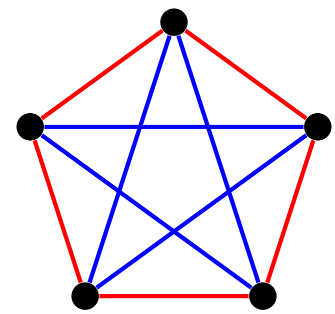
- In order to prove that $R(m,n) \geq N$ for some integer N, you must somehow construct a red,blue-coloring of $E(K_{N-1})$ which has neither a red copy of K_m nor a blue copy of K_n.

- In order to prove that $R(m,n) \leq N$ for some integer N, you must show that *every* red,blue-coloring of $E(K_N)$ contains either a red copy of K_m or a blue copy of K_n.

So the philosophy behind lower bounds vs upper bounds is very different.

Theorem 3. $R(3,3) = 6$.

Proof. We need to prove two inequalities; we start by showing that $R(3,3) \geq 6$, so we must construct a red,blue-coloring of $E(K_5)$ which has no monochromatic triangle. The following is such a coloring:



Indeed, both the red-graph and the blue-graph are copies of C_5, which is triangle-free. In fact, it can be shown that this is the *only* coloring of $E(K_5)$ (up to permuting the vertices) which has no monochromatic triangle.

Now we need to prove that $R(3,3) \leq 6$, so we need to show that *every* red,blue-coloring of $E(K_6)$ has a monochromatic triangle. So fix any red,blue-coloring of $E(K_6)$. Fix any vertex v and let R be the red-neighborhood of v and let B be the blue-neighborhood of v, so $N(v) = R \sqcup B$. Since $|N(v)| = 5$, we have either $|R| \geq 3$ or $|B| \geq 3$. We are simply looking for a monochromatic triangle, so the colors are symmetric for us; thus without loss of generality, suppose that $|R| \geq 3$. Now, if any edge $xy \in \binom{R}{2}$ is red, then $\{v, x, y\}$ induces a red copy of K_3. Otherwise, every edge in $\binom{R}{2}$ is blue, and so R contains a blue copy of K_3 since $|R| \geq 3$. □

The same idea used to prove that $R(3,3) \leq 6$ can be exploited to bound $R(m,n)$ in general.

Theorem 4. *For integers $m, n \geq 2$, if $R(m-1,n)$ and $R(m,n-1)$ both exist, then*

$$R(m,n) \leq R(m-1,n) + R(m,n-1).$$

In particular, $R(m,n)$ exists as well.

Proof. Let $N = R(m-1,n) + R(n,m-1)$ and let f be any red,blue-coloring of $E(K_N)$. We claim that either f contains a red copy of K_m or a blue copy of K_n, which will imply that $R(m,n) \leq N$ (and thus exists) as claimed.

Fix any vertex v, let R be the red-neighborhood of v and let B be the blue-neighborhood of v. We know that $|R| + |B| = N - 1$ and so either $|R| \geq R(m-1,n)$ or $|B| \geq R(m,n-1)$ by the definition of N.

Suppose first that $|R| \geq R(m-1,n)$ and consider restricting the coloring of $E(K_N)$ to $\binom{R}{2}$. Since $|R| \geq R(m-1,n)$, this restricted coloring contains either a red copy of K_{m-1} or a blue copy of K_n. If it has a blue copy of K_n, then we're done! Otherwise, let $R' \subseteq R$ be the vertices of a red copy of K_{m-1}. Since every edge between v and R is red, we then have that $R' \cup \{v\}$ induces a red copy of K_m as needed.

If $|B| \geq R(m,n-1)$, then the argument is identical, just with the colors flipped. □

Corollary 5. *For any positive integers m, n, the Ramsey number $R(m,n)$ exists and*

$$R(m,n) \leq \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}.$$

We'll need something known as Pascal's identity: for integers $n \geq k \geq 1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

If you've never seen this before, it's a good exercise! Basically, partition $\binom{[n]}{k}$ into those sets which contain the element n and those sets which don't.

Proof. We prove this double-induction on m and n.

If $m = 1$ or $n = 1$, then certainly $R(m,n) = 1 = \binom{m+n-2}{0}$ (since we just need a single vertex).

We may thus suppose that $m, n \geq 2$. The induction hypothesis implies that $R(m-1,n)$ and $R(m,n-1)$ both exist, so Theorem 4 implies that $R(m,n)$ also exists and (also by the induction hypothesis)

$$R(m,n) \leq R(m-1,n) + R(m,n-1) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}. \qquad \square$$

Corollary 6. *For any positive integer n, we have*

$$R(n,n) \leq \binom{2n-2}{n-1} \leq 2^{2n-2} \leq 4^n.$$

To date, $R(n,n) \leq 4^n$ is essentially the best-known upper bound for large values of n. Only very, very minor improvements have been made, but no-one has been able to prove any bound of the form $R(n,n) \leq (3.9999)^n$.

Corollary 7. *Any red,blue-coloring of $E(K_n)$ contains a monochromatic clique on $> \frac{1}{2}\log_2 n$ many vertices.*

Stated in graph land: Any n-vertex graph G has either $\omega(G) > \frac{1}{2}\log_2 n$ or $\alpha(G) > \frac{1}{2}\log_2 n$.

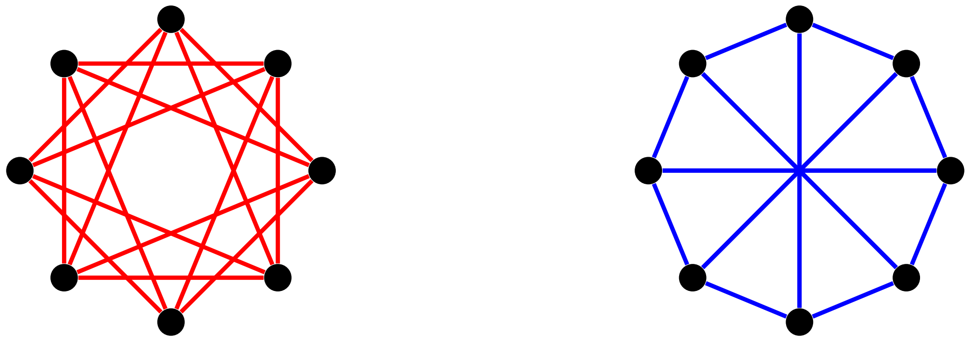
Proof. Let k be the largest integer for which $2^{2k-2} \leq n$. Since $R(k,k) \leq 2^{2k-2} \leq n$, we know that any red,blue-coloring of $E(K_n)$ contains a monochromatic copy of K_k. Now, by the definition of k, we know that

$$2^{2(k+1)-2} > n \implies 2(k+1) - 2 > \log_2 n \implies k > \frac{1}{2}\log_2 n. \qquad \square$$

From Theorem 4, we have $R(4,3) \leq R(3,3) + R(4,2) = 6 + 4 = 10$. It turns out that the real answer is 9. This can be proved by a slight modification of the proof of Theorem 4.

Theorem 8. *$R(4,3) = 9$.*

Proof. Lower bound: In order to prove that $R(4,3) \geq 9$, we need to show that there is a red,blue-coloring of $E(K_8)$ which does not contain a red copy of K_4 nor a blue copy of K_3. Below is such a coloring (we draw the red and blue graphs separately to help with visibility):



The actual coloring is thus: for $i \neq j \in \{0, \ldots, 7\}$, color the edge ij blue if $i - j \in \{\pm 1, 4\} \pmod 8$ and otherwise color ij red. I'll leave it to you to check that this coloring actually works :)

Upper bound: In order to show that $R(4,3) \leq 9$, we need to show that *any* red,blue-coloring of $E(K_9)$ contains either a red copy of K_4 or a blue copy of K_3. Indeed, fix any red,blue-coloring of $E(K_9)$ and fix a vertex v. Let R, B denote the red- and blue-neighborhoods of v, respectively. By exactly the same logic used above, we win if either $|R| \geq R(3,3) = 6$ or if $|B| \geq R(4,2) = 4$. Thus, suppose that $|R| \leq 5$ and $|B| \leq 3$. Since $|R| + |B| = |N(v)| = 8$, this implies that $|R| = 5$ and $|B| = 3$. Since v was arbitrary, this holds for every vertex v.

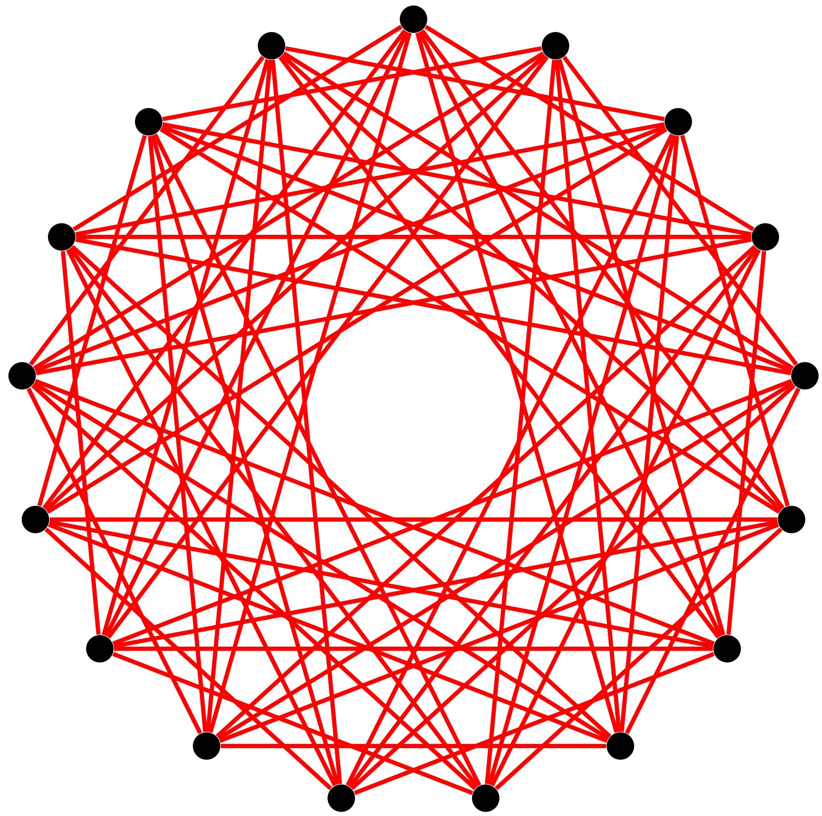
Now, let G_r denote the red-graph. By what we just said, G_r is 5-regular. However, G_r has 9 vertices, and so G_r has an odd number of odd degrees; a contradiction. □

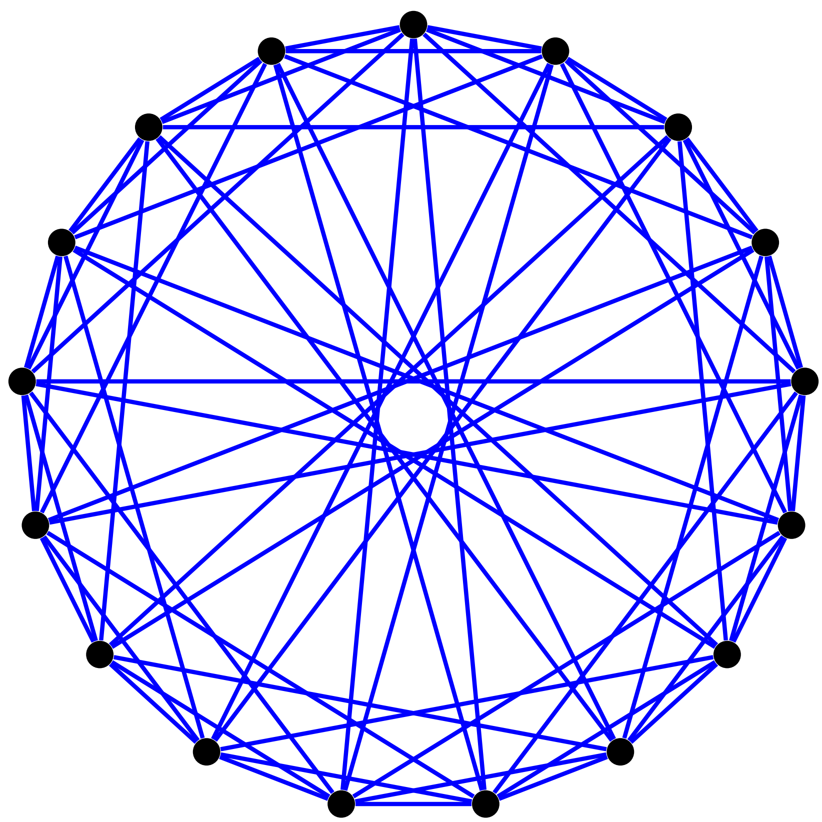
Now for $R(4,4)$.

Theorem 9. $R(4,4) = 18$

Proof. For the upper bound, we have $R(4,4) \leq R(3,4) + R(4,3) = 18$.

For the lower bound, the following is a red,blue-coloring of $E(K_{17})$ which avoids monochromatic copies of K_4 (provided I didn't make a mistake drawing it). The definition of the coloring is thus: for $i \neq j \in \{0, \ldots, 16\}$, color the edge ij blue if $i - j \in \{\pm 1, \pm 2, \pm 4, \pm 8\} \pmod{17}$ and otherwise color ij red. I'll leave it to you to check that this coloring actually works :) If you actually want to understand why this coloring works, look up "quadratic residues" and "Payley graphs".





□

As far as I'm aware, the best-known bounds for $R(5,5)$ are $43 \leq R(5,5) \leq 48$ and the best-known bounds for $R(6,6)$ are $102 \leq R(6,6) \leq 165$.

The following is a quote from Paul Erdős, who helped to develop the entire study of Ramsey theory:

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Let's end today with a fun application of Ramsey numbers.

Let $(x_1, \ldots, x_n)$ be a sequence of real numbers. The *length* of this sequence is n. The sequence is said to be *increasing* if $x_1 < \cdots < x_n$ and is said to be *decreasing* if $x_1 > \cdots > x_n$. The sequence is said to be *monotone* if it is either increasing or decreasing. Note that (x) is both increasing and decreasing.

A *subsequence* of a sequence is formed by deleting some elements from the original sequence. For example, $(2,3,4,1,7)$ is a subsequence of $(10, \underline{2}, 8, 9, \underline{3}, -20, \underline{4}, \underline{1}, 30, 12, \underline{7}, -3, -4)$. Formally, if $(x_1, \ldots, x_n)$ is a sequence, then a subsequence is any sequence of the form $(x_{i_1}, \ldots, x_{i_k})$ where $i_1 < \cdots < i_k \in [n]$. Note that (x_i) is a subsequence of $(x_1, \ldots, x_n)$ for any $i \in [n]$.

Reasonably, if $(x_1, \ldots, x_n)$ is a sequence of real numbers, then a *monotone subsequence* of $(x_1, \ldots, x_n)$ is a subsequence of $(x_1, \ldots, x_n)$ which is monotone. Of course, if $(x_1, \ldots, x_n)$ is already monotone, then any subsequence is also monotone.

Theorem 10 (Erdős–Szekeres)**.** *Every sequence of n* distinct *real numbers contains a monotone subsequence of length $> \frac{1}{2}\log_2 n$.*

Really, the Erdős–Szekeres theorem tells us that any sequence of n distinct real numbers contains a monotone subsequence of length $\geq \sqrt{n}$, which is actually optimal. This can be proved through a similar argument to what follows, but invoking DS5.3.4 instead of what we've shown about Ramsey numbers.

Proof. Let $(x_1, \ldots, x_n)$ be any sequence of n distinct real numbers

Create a red,blue-coloring of $E(K_n)$ (think of $V(K_n) = [n]$) where, for $i < j \in [n]$, we color the edge ij red if $x_i < x_j$ and color the edge ij blue if $x_i > x_j$. Since the x_i's are distinct, every edge gets a color.

Now, Corollary 7 tells us that this coloring contains a monochromatic copy of K_k for some $k > \frac{1}{2}\log_2 n$; label the vertices of such a K_k as $i_1 < i_2 < \cdots < i_k$.

If this is a red copy of K_k, then we find that $(x_{i_1}, \ldots, x_{i_k})$ is an increasing subsequence. If this is a blue copy of K_k, then we find that $(x_{i_1}, \ldots, x_{i_k})$ is a decreasing subsequence. In either case, we have located a monochromatic subsequence of length $k > \frac{1}{2}\log_2 n$. $\square$