These notes are from https://mathematicaster.org/teaching/graphs2022/extra_04-19.pdf

Recall that last time we proved that if G is a connected, planar graph on $n \geq 3$ vertices, then $|E(G)| \leq 3n - 6$. Let's use this fact to prove the 6-color theorem. We begin by showing that the minimum degree of any planar graph is small.

Lemma 1. *If G is a planar graph, then $\delta(G) \leq 5$.*

Proof. If G has connected components $G_1, \ldots, G_k$, then $\delta(G) = \min_{i \in [k]} \delta(G_i)$, so it suffices to consider the case when G is connected. Set $n = |V(G)|$. If $n \leq 2$, then $\delta(G) \leq 1$, so we may suppose that $n \geq 3$. Thus, we can apply Theorem 9 from last lecture along with the handshaking lemma to bound.

$$3n - 6 \geq |E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg v \geq \frac{1}{2} \sum_{v \in V(G)} \delta(G) = \frac{n}{2}\delta(G) \implies \delta(G) \leq \frac{2}{n}(3n-6) = 6 - \frac{12}{n} < 6.$$

Since $\delta(G)$ and 6 are integers, we conclude that $\delta(G) \leq 5$ which concludes the proof. □

Corollary 2 (6-color theorem)**.** *If G is a planar graph, then $\chi(G) \leq 6$.*

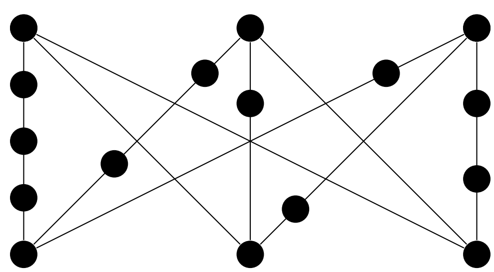
Proof. If H is a subgraph of G, then H is also planar. Therefore, thanks to Lemma 1

$$d(G) = \max\{\delta(H) : H \text{ is a subgraph of } G\} \leq 5.$$

We may thus use our greedy-coloring bound to see that $\chi(G) \leq d(G) + 1 \leq 6$. □

We'll return to coloring planar graphs in just a moment, but now for something somewhat different.

We ended last lecture by showing that K_5 and $K_{3,3}$ are indeed non-planar graphs. It turns out that K_5 and $K_{3,3}$ are "morally" the only non-planar graphs. What does "morally" mean here? Well, if G is planar, than any subgraph is planar; contrapositively, if G contains any non-planar subgraph, then G is also non-planar. So G is non-planar if it contains a copy of K_5 or $K_{3,3}$. But consider the following graph where we just "place some extra vertices" on some of the edges of $K_{3,3}$:



Certainly this graph isn't planar, but it does not contain a copy of $K_{3,3}$ (nor of K_5), as much as it "looks like" $K_{3,3}$.

Definition 3. *Let G be a graph. A* subdivision *of G is formed by replacing some of the edges of G by paths (each connecting the original vertices of the edge that it replaced) where these paths are internally disjoint.*

Note that G is a subdivision of itself (since we can simply do nothing). Also, if H is a subdivision of G and G is a subdivision of J, then H is a subdivision of J. Also, note that every "new" vertex introduced when creating a subdivision has degree exactly 2.

Observation 4. *Let G be a graph and let H be any subdivision of G. Then G is planar if and only if H is planar.*

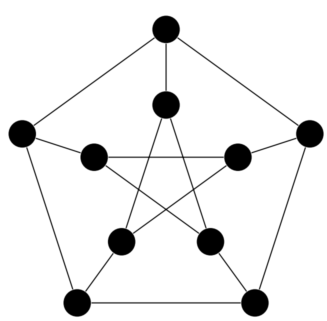
Indeed, given a planar drawing of G, we can just place down some extra vertices on some of the drawn edges of G to get a planar drawing of H. In the other direction, given a planar drawing of H, we can just follow the curve traced out by each of these extra paths (and forget that there were vertices on it) to get a planar drawing of G. This works precisely because each of the vertices introduced when creating H has degree exactly 2.

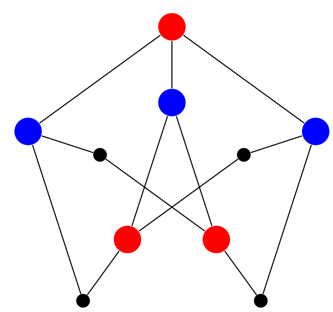
Therefore:

Observation 5. *If G contains a subdivision of either K_5 or of $K_{3,3}$, then G is non-planar.*

Now, for some reason that I honestly can't understand, graph theorists are obsessed with a particular graph known as the Petersen graph (pictured below). If you've read the book, you've probably seen it mentioned a couple times. The Petersen graph is pretty, but beyond that, I haven't found any good reason to actually show it to you before now. However, they'd probably revoke my discrete-math-card if I let you leave this class without having ever seen it, so here we go!

The Petersen graph is non-planar. This probably isn't too surprising — it looks so much like K_5! But how do you actually prove that it's non-planar? The picture on the right displays a subdivision of $K_{3,3}$ contained within the Petersen graph and so it cannot be planar. Interestingly, as much as the Petersen graph resembles K_5, it cannot contain a subdivision of K_5 since the Petersen graph is 3-regular yet any subdivision of K_5 has 5 vertices of degree 4.[1]





Okay, great, if we can find a copy of a subdivided K_5 or $K_{3,3}$, then we know the graph is non-planar. This doesn't seem too terribly interesting... except!

Theorem 6 (Kuratowski's Theorem)**.** *G is* non-*planar if* ***and only if*** *G contains a copy of some subdivision of either K_5 or $K_{3,3}$.*

We won't give a proof of Kuratowski's theorem, but I recommend looking one up if you feel so inclined. Also, there are other similar classifications of planar graphs (using more general structures than subdivisions), e.g. Wagner's theorem.

Now back to coloring! Any excuse to break out my box of crayons and feel like a 5-year-old again!

[1]There is, however, a sense in which the Petersen graph "contains a copy" of K_5, known as minors (look up Wagner's theorem if you're interested).

We end today (along with our excursion into planar graphs) by proving the 5-color theorem (so we're only one color away from the famous 4-color theorem!). In order to do so, we will rely on a technique known as Kempe swaps, which is a general idea and has nothing to do with planar graphs (though an attempted proof of the 4-color theorem is the origin of this idea).

Let G be a graph and let $f\colon V(G) \to C$ be a proper coloring of G where C is some set of colors. Fix any two colors $a \neq b \in C$ and consider the subgraph of G induced by the vertices which receive either color a or color b. An a, b-Kempe component of the coloring f is any connected subgraph of this a, b-colored induced subgraph of G. Fix any such a, b-Kempe component $C_{a,b}$ of f. The Kempe swap on f applied to $C_{a,b}$ is performed by switching the colors a and b within $C_{a,b}$ (which has only the colors a and b). Formally, the Kempe swap on f applied to $C_{a,b}$ is a new coloring $f'\colon V(G) \to C$ where

$$f'(v) = \begin{cases} f(v) & \text{if } v \notin V(C_{a,b}), \\ a & \text{if } v \in V(C_{a,b}) \text{ and } f(v) = b, \\ b & \text{if } v \in V(C_{a,b}) \text{ and } f(v) = a. \end{cases}$$

Observe that f' is still a proper coloring of G. Indeed, f' does not change the colors outside of $V(C_{a,b})$ and maintains a proper coloring within $V(C_{a,b})$ (since we just switched the colors of the vertices on each edge and $C_{a,b}$ is an induced subgraph of G). Furthermore, since $C_{a,b}$ was a connected component of the a, b-colored induced subgraph of G, so if $xy \in E(G)$ with $x \in V(C_{a,b})$ and $y \notin V(C_{a,b})$, then f assigns y a color other than a, b.

Kempe swaps are useful for "extending a coloring". Suppose we've properly colored some vertices of our graph, but we sill need to color more. However, some uncolored vertex is adjacent to vertices which see all available colors... Instead of giving up and going home, we can attempt to perform a sequence of Kempe swaps on the already-colored vertices to "free up" a color that we can then use to continue our coloring.

Theorem 7 (5-color theorem)**.** *If G is a planar graph, then $\chi(G) \leq 5$.*

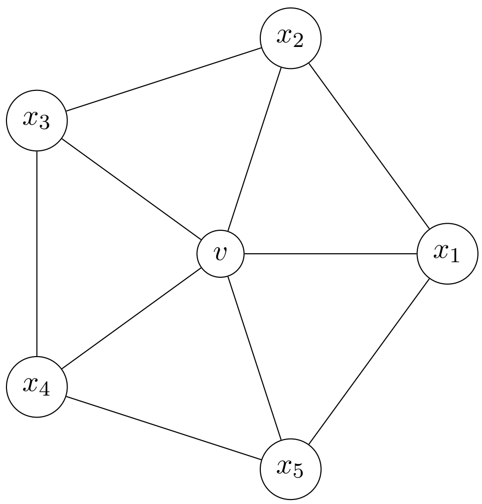
Proof. Suppose the claim is false, so $\chi(G) \geq 6$. Since we can pass to a 6-critical subgraph and every subgraph of G is also planar, we may suppose that G itself is 6-critical without loss of generality. Applying some facts we proved two weeks ago, we know that $\delta(G) \geq 5$. Additionally, $\delta(G) \leq 5$ since G is planar (Lemma 1), so actually $\delta(G) = 5$. Fix any $v \in V(G)$ with $\deg v = 5$.

Now, G is 6-critical, so we know that $\chi(G - v) \leq 5$, so let $f\colon V(G) \setminus \{v\} \to [5]$ be a proper coloring of $G - v$. Now, if some color is un-unsed in $N(v)$, then we can give v that color to arrive at a proper 5-coloring of G; a contradiction. Therefore, the 5 neighbors of v each receive a unique color. Our goal is to perform some Kempe swaps on f to "free up" one of these colors to then allow coloring v, or else somehow conclude that G is non-planar.

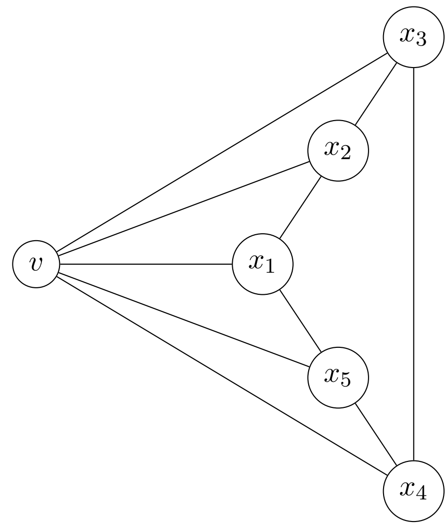
Now, G is planar, so fix some planar embedding of G; hence, we will treat G as a plane graph from now on. In this embedding, look at how the 5 edges incident to v leave v and label the neighbors of v as $x_1, \ldots, x_5$ in anticlockwise order (starting wherever you please) based on how the corresponding edge leaves v (see the picture(s) below). Technically, the actual vertices may not actually appear to be in anticlockwise, but the important thing is that the corresponding edges are. By permuting the colors if necessary, we may suppose that $f(x_i) = i$.

We next claim that we may suppose that $x_i x_{i+1}$ is an edge of G for each $i \in [5]$, where $x_{5+1} = x_1$. Indeed, based on the ordering, we know that x_i and x_{i+1} are incident to a common face. If x_i and x_{i+1} are not adjacent, then we can draw a curve connecting x_i to x_{i+1} which lives entirely within this face (this technically requires a bit of an argument beyond us at the moment); thus, adding this edge to the plane graph G is still a plane graph. Note that by adding these extra edges (should they not already exist), G may no longer 6-critical, so we can no longer rely on this fact; however, f is still a proper coloring of this modified graph (sans v) since none of $x_1, \ldots, x_5$ receive the same color. Thus, we still reach a contradiction if we can somehow color v.

Now, looking only at $v, x_1, \ldots, x_5$, the drawing looks something like one of the following pictures (pictures are misleading, though):



or



It is technically possible to argue that the left picture can be assumed, but we really shouldn't rely on pictures in our proof. But keeping a picture in your head is helpful, and the left picture will give you a good enough intuition. Additionally, there could be extra edges between the x_i's and there certainly will be edges between the x_i's and other vertices not drawn in the picture, but v has no additional edges.

Now, consider the $1,3$-Kempe component of f which contains x_1. If x_3 does not reside in this Kempe component, then we may perform a Kempe swap on this component to recolor x_1 with color 3 and not affect any other colors among $x_2, \ldots, x_5$. Thus, we can now color v with color 1 to get a proper coloring of G; a contradiction. Thus, x_3 must reside within this Kempe component; in particular, there is a path from x_1 to x_3 which uses only the colors 1 and 3; call this path $P_{1,3}$. Of course, $P_{1,3}$ contains no vertices from $v, x_1, \ldots, x_5$ except for x_1 and x_3.

Next, consider the $2,4$-Kempe component of f which contains x_2. If x_4 does not reside in this Kempe component, then we may similarly perform a Kempe swap on this component to recolor x_2 with color 4 and then color v with color 2 to get a proper coloring of G; a contradiction. Thus, by the same reasoning as above, there is a path from x_2 to x_4 which uses only the colors 2 and 4; call this path $P_{2,4}$. Again, $P_{2,4}$ contains no vertices from $v, x_1, \ldots, x_5$ except for x_2 and x_4. Furthermore, $P_{1,3}$ and $P_{2,4}$ are vertex-disjoint since these sets of colors are disjoint.

Now, if you look at the pictures above, it's pretty easy to convince yourself that this is impossible since $P_{1,3}$ and $P_{2,4}$ must cross at some edge (since they can't share a vertex), but we really shouldn't rely on pictures since they're very misleading. Instead, we show that we have found a subdivision of K_5; a contradiction since we know G is planar.

Indeed, consider the vertices v, x_1, x_2, x_3, x_4. The only edges that we're missing to get a K_5 are x_1x_3, x_2x_4 and x_4x_1. Now, $P_{1,3}$ is an x_1-x_3 path, $P_{2,4}$ is an x_2-x_4 path and (x_4, x_5, x_1) is an x_4-x_1 path. These paths are internally disjoint and have no internal vertices within $v, x_1, \ldots, x_4$; thus, we have found a subdivision of K_5 within G; a contradiction. $\square$

Question: What was color 5 doing in the above proof? We never really touched it and didn't even wlog based on it. Why couldn't we perform essentially the same proof with just 4 colors and thus have a proof of the 4-color theorem? Well, Alfred Kempe (after whom Kempe swaps are named) certainly thought so and published a proof of the 4-color theorem along these lines in 1879. It was so convincing that it took 11 years before someone noticed the error! Then it took until 1976 before we had a real proof of the 4-color theorem (provided you trust computers[2]). Even today, we still don't have a "satisfactory" proof of the 4-color theorem.

If you have the time, try to repeat the argument with only 4 colors instead of 5. Here, you will also have to handle the case when $\delta(G) = 4$, but this turns out to be fairly simple (essentially identical to our proof above). It's the case when $\delta(G) = 5$ and one color is repeated in v's neighborhood that causes the issue. I found the following notes online `https://web.math.ucsb.edu/~padraic/ucsb_2014_15/math_honors_f2014/math_honors_f2014_lecture4.pdf` which describes Kempe's "proof" of the 4-color theorem (and has really pretty pictures that I'm incapable of drawing); see if you can spot the error! In my opinion, the error mostly boils down to "pictures can be very misleading".

I found the following quote from John Conway (inventor of Conway's game of life if you've heard of that) about Kempe's "proof":

> I did read a few of the papers from this period, including Kempe's proof, Tait's deduction of his edge-coloring criterion from it, and an article in which Heawood pointed out the mistake, among other things, and the impression I got from them was much the same as Jim's. There was indeed a fair amount of interest in Kempe's "theorem", but not much evidence that any great number of people actually scrutinized his proof, or even read it. I don't think it would have made much difference if they had. The proof is very convincing, and in the days when the amateur provers were still interested in FCT, 2 times out of 3 "Kempe's Catastrophe", as Tom O'Beirne used to call it, was the proof they'd produce. Tom had a set lecture in which he gave Kempe's proof, illustrated by a specially made-up board with colored pegs, and seldom could anyone in the audience find anything wrong with it.

[2]You youngins out there may reply "Computers are great! Why would you trust humans? After all, Kempe's proof was wrong.". Fair point, but math, at its heart, is about *why* something is true, not just *that* it's true. Even though Kempe's proof was wrong, the ideas contained therein are crucial to the current proof that we have and have been applied in numerous other settings.