These notes are from https://mathematicaster.org/teaching/graphs2022/extra_04-14.pdf

Let's start with a motivating problem: There are three houses each wanting to be connected to three utilities (maybe water, gas an electric). For some reason, these utilities need to run individual connections to each of these houses. Furthermore, they would prefer that one of these utility lines "criss-cross" to avoid any interference. Can this be done?

Phrasing this in graph-theory land, we are asking if we can draw the graph $K_{3,3}$ in the plane so that no pair of edges cross one another... By the end of this lecture, we will have an answer.

Informally speaking, a *planar graph* is a graph that can be drawn in the plane without any crossing edges.[1] Such a drawing is called a *plane graph*.

To stress this distinction: a plane graph a graph which *is* drawn (validly) in the plane; a planar graph is a graph which *can be* drawn (validly) in the plane. When dealing with a planar graph, usually the first step is to turn it into a plane graph by fixing some drawing.

This distinction is very important. The following two graphs are the same planar graph (literally, not even just up to isomorphism), but are not the same plane graph based on how we drew them.

Unfortunately, we don't have the background at the moment to go into the formalism behind a number of facts (and even some basic definitions) about planar/plane graphs. To do so properly, we would need some topology know-how, particularly the Jordan Curve Theorem. So, there are a number of facts that I will throw at you without formal justification, but I will give you the intuition behind them.

One fact that we'll use implicitly when drawing planar graphs (but not actually rely on for any proof) is the following:

Fact 1. *If G is a planar graph, then we can find a planar drawing of G wherein each edge is a line segment.*

That is, our edges don't have to be all curvy and crazy! (though they can be as curvy and crazy as we wish should we desire that). Perhaps I'll be able to justify this fact later, but it isn't crucial to our understanding; it just helps us visualize things.

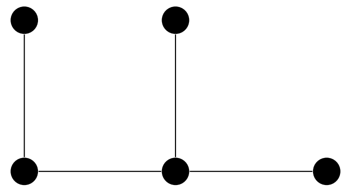
What is important is the fact that a plane graph "cuts out" regions of the plane.

Definition 2. *The plane graph G breaks the plane into a bunch of connected regions. We call these regions* faces *of G. Furthermore, there is exactly one "unbounded" face which we refer to as the* exterior face*. All other faces, should they exist, are called* interior faces*.*

We denote the set of faces of G by $F(G)$.

Note that if we draw a tree in the plane, then there is only one face: the exterior face:

[1] If you want to see the formalism behind this, look at the last page of these notes.



A vertex/edge is said to be *incident* to a face if that vertex/edge lives on the boundary of that face. Note that a face could be incident to many vertices, a vertex could be incident to many faces, but an edge can be incident to at most two faces. Reasonably, each vertex/edge is incident to at least one face. A face is always incident to at least one vertex (if we don't consider the null-graph to be a thing) and (unless the graph has no edges) it is always incident to at least one edge.

With this lingo we can actually see pretty clearly that the two plane graphs drawn above are not the same. Indeed, the plane graph on the right has two faces which have 4 incident edges while the plane graph on the left has no such faces. However, they do both have exactly 4 faces; this is no coincidence as we will see later.

Fact 3. *Let G be a plane graph. An edge e is incident to exactly one face of G if and only if e is a bridge.*

I can only give you an intuition about this proof right now. Essentially, if e is incident to only one face, then, upon deleting that edge, the incident face now wraps all the way around some chunk of vertices of G. No edge can cross a face, and so this face "realizes" a break in G. On the other hand, if e is incident to two faces, then "walking around" one of these faces will yield a walk between the vertices of e which does not actually use the edge e, indicating that e is part of a cycle.

We now have a great theorem, which relates the number of vertices, edges and faces of a (connected) plane graph.

Theorem 4 (Euler's Formula)**.** *If G is any connected plane graph, then*

$$|V(G)| + |F(G)| - |E(G)| = 2.$$

Proof. We prove this by induction on $m = |E(G)|$. Set $n = |V(G)|$ for convenience. Since G is connected, we must have $m \geq n-1$.

If $m = n-1$, then G is a tree (since it is connected). Thus, G has exactly one face (the exterior face), so $|F(G)| = 1$. As such, $n + |F(G)| - m = n + 1 - (n-1) = 2$ as claimed.

Thus suppose that $m \geq n$, so we know that G contains a cycle C. Let $e \in E(C)$ be any edge of this cycle, so e is not a bridge, and set $H = G - e$. Of course, H is still a plane graph (we just inherit the drawing of G).

Since e was not a bridge, we know that H is connected and also that e was incident to two faces in G. Upon removing e, these two faces have become one (and no other faces have changed). Thus, $|F(H)| = |F(G)| - 1$. Additionally, $|E(H)| = m - 1$ and $|V(H)| = n$. Thus, by the induction hypothesis, we have

$$n + |F(G)| - m = |V(H)| + (|F(H)| + 1) - (|E(H)| - 1) = |V(H)| + |F(H)| - |E(H)| = 2. \quad \square$$

Note that connectivity is crucial in Euler's formula. However, if we started with forests as our base-case instead of trees, we could have proved the following:

Theorem 5 (Euler's Formula for disconnected graphs)**.** *Let G be a plane graph and let* $\mathrm{comp}(G)$ *denote the number of connected components of G. Then,*

$$|V(G)| + |F(G)| - |E(G)| = 1 + \mathrm{comp}(G).$$

Generally, we will work only with connected plane graphs, so we won't have to worry about this.

Corollary 6. *If G is any planar graph, then any two drawings of G in the plane have the same number of faces.*

In other words, while faces are not an inherent property of a planar graph, the number of faces (upon drawing it) is.

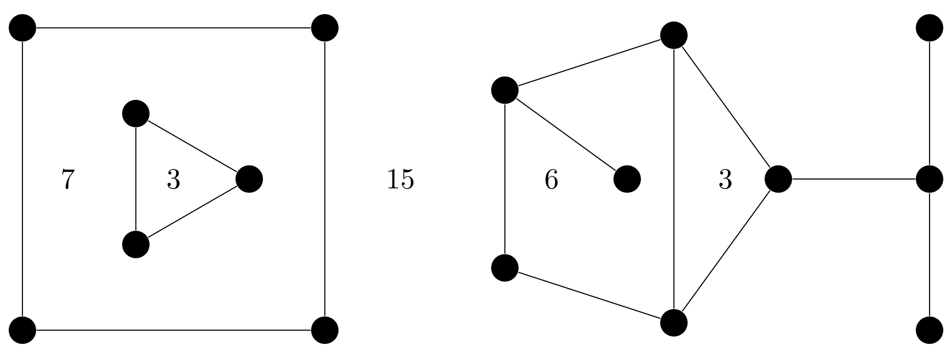
Let's think a bit more about these faces. Each face has a bunch of edges on its boundary (unless G is just a bunch of isolated vertices). Consider "walking around" the face and counting every edge you encounter. We call this count the *length of the face* with one **very important caveat**: If the face is incident to a bridge, than the bridge is counted twice since you needed to cross that edge twice to get all the way around.

Definition 7. *Let G be a plane graph and let $f \in F(G)$ be a face of G. The* length *of f, denoted by* $\operatorname{len}(f)$*, is defined to be*

$$\operatorname{len}(f) = \#\{\textit{edges incident to } f\} + \#\{\textit{bridges incident to } f\}.$$

Adding on the number of bridges incident to f encapsulates the idea that any bridge must be crossed twice to get around the face. Oftentimes, the plane graph won't have any bridges, so we won't need to worry about this caveat.

As an example, in the tree drawn earlier, the exterior face has length 8, even though it is only incident to 4 edges. As another example, here is a (disconnected) plane graph where each face is labeled with its length.



Let's now introduce a version of the handshaking lemma which relates face-lengths and edges:

Lemma 8 (Headshaking Lemma)**.** *If G is a plane graph, then*

$$\sum_{f \in F(G)} \operatorname{len}(f) = 2|E(G)|.$$

Before diving into the proof, we introduce a useful notion known as an indicator function. For a proposition P, we write $\mathbf{1}[P]$ which is 1 if P is true and 0 if P is false. Suppose that Ω is some finite set and that $X \subseteq \Omega$. The key observation is that

$$|X| = \sum_{x \in \Omega} \mathbf{1}[x \in X].$$

In other words, we go to every element of Ω and ask "are you in X?", and then add up all the "yes"s.

Proof. Suppose that $G = (V, E)$ and set $F = F(G)$ for convenience. Also, let $B \subseteq E$ denote the set of bridges of G. For $e \in E$, define F_e to be the set of faces incident to e. Note that

$$|F_e| = \begin{cases} 1 & \text{if } e \in B, \\ 2 & \text{otherwise.} \end{cases}$$

For $f \in F$, define E_f to be the set of edges incident to f and define B_f to be the set of bridges incident to f. By definition, $\text{len}(f) = |E_f| + |B_f|$. Note that $B_f \subseteq E_f$ and that $e \in E_f \iff f \in F_e$.

We now write the following string of equalities by exploiting indicator functions and switching the order of summation.

$$\begin{aligned}
\sum_{f \in F} \text{len}(f) &= \sum_{f \in F} \big(|E_f| + |B_f|\big) = \sum_{f \in F} \sum_{e \in E} \big(\mathbf{1}[e \in E_f] + \mathbf{1}[e \in B_f]\big) \\
&= \sum_{e \in E} \sum_{f \in F} \big(\mathbf{1}[e \in E_f] + \mathbf{1}[e \in B_f]\big) \\
&= \sum_{e \in B} \sum_{f \in F} \big(\mathbf{1}[e \in E_f] + \mathbf{1}[e \in B_f]\big) + \sum_{e \in E \setminus B} \sum_{f \in F} \big(\mathbf{1}[e \in E_f] + \mathbf{1}[e \in B_f]\big)
\end{aligned}$$

Now, if $e \notin B$, then $e \notin B_f$ for any $f \in F$ and $e \in E_f \iff f \in F_e$; thus $\mathbf{1}[e \in E_f] + \mathbf{1}[e \in B_f] = \mathbf{1}[f \in F_e]$ if $e \in E \setminus B$. Additionally, if $e \in B$, then $e \in E_f \iff e \in B_f \iff f \in F_e$; thus $\mathbf{1}[e \in E_f] + \mathbf{1}[e \in B_f] = 2 \cdot \mathbf{1}[f \in F_e]$ if $e \in B$. We can therefore continue our string of equalities:

$$\begin{aligned}
\sum_{f \in F} \text{len}(f) &= \sum_{e \in B} \sum_{f \in F} 2 \cdot \mathbf{1}[f \in F_e] + \sum_{e \in E \setminus B} \sum_{f \in F} \mathbf{1}[f \in F_e] \\
&= \sum_{e \in B} 2|F_e| + \sum_{e \in E \setminus B} |F_e| = \sum_{e \in B} 2 + \sum_{e \in E \setminus B} 2 \\
&= 2|B| + 2|E \setminus B| = 2|E|.
\end{aligned}$$

□

A similar proof would have to be performed in order to establish a handshaking lemma for "pseudographs" (multigraphs which can have loops). This is not a coincidence: there is a natural correspondence between plane graphs and pseudographs (known as dual graphs), but we won't get into this now (we have too many other fun things to cover!).

Intuitively, a planar graph cannot have "too many" edges.

Theorem 9. *Let G be a connected planar graph on n vertices. If $n \geq 3$, then*

$$|E(G)| \leq 3n - 6.$$

The connectivity assumption can be removed, but it requires further argument to do so. Also, the theorem is false if $n \leq 2$.

Proof. Since G is a planar graph, we can embed G is the plane, so treat G as a plane graph throughout the proof.

Observe that G cannot contain any faces of lengths 1 or 2. Indeed a face of length 1 is always impossible, and a face of length 2 would have a single, disjoint edge (i.e. a connected component isomorphic to K_2) as its boundary. The fact that G is connected and has at least 3 vertices rules the latter out.

Therefore, $\text{len}(f) \geq 3$ for all $f \in F(G)$. Using the headshaking lemma, we thus bound

$$2|E(G)| = \sum_{f \in F(G)} \text{len}(f) \geq \sum_{f \in F(G)} 3 = 3|F(G)| \implies |F(G)| \leq \frac{2}{3}|E(G)|.$$

Now, G is a connected plane graph, so Euler's formula tells us that

$$2 = n + |F(G)| - |E(G)| \leq n + \frac{2}{3}|E(G)| - |E(G)| = n - \frac{1}{3}|E(G)| \implies |E(G)| \leq 3(n-2). \quad \square$$

We can now encounter our first non-planar graph!

Proposition 10. *K_5 is* not *a planar graph.*

Proof. K_5 has $5 \geq 3$ vertices and $\binom{5}{2} = 10$ edges. Since K_5 is connected, if K_5 happened to be planar, then Theorem 9 would tell us that $10 \leq 3 \cdot 5 - 6 = 9$; a contradiction. $\square$

There is one other really important non-planar graph: $K_{3,3}$. $K_{3,3}$ has 6 vertices and 9 edges; since $9 \leq 12 = 3 \cdot 6 - 6$, the same proof won't work here. Instead, we need to somehow take into account the fact that $K_{3,3}$ is bipartite.

Theorem 11. *Let G be a connected planar graph on n vertices. If $n \geq 3$ and G is triangle-free, then*

$$|E(G)| \leq 2n - 4.$$

The connectivity assumption can be removed, but it requires further argument to do so. Also, the theorem is false if $n \leq 2$.

Proof. Since G is a planar graph, we can embed G in the plane, so treat G as a plane graph throughout the proof.

Just as above, G cannot contain any faces of length 1 or 2. Additionally, a face of length 3 would yield a triangle in G (why?). Thus, $\text{len}(f) \geq 4$ for all $f \in F(G)$. Using the headshaking lemma, we thus bound

$$2|E(G)| = \sum_{f \in F(G)} \text{len}(f) \geq 4|F(G)| \implies |F(G)| \leq \frac{1}{2}|E(G)|.$$

Finally, G is a connected plane graph, so we can apply Euler's formula to find

$$2 = n + |F(G)| - |E(G)| \leq n + \frac{1}{2}|E(G)| - |E(G)| = n - \frac{1}{2}|E(G)| \implies |E(G)| \leq 2(n-2). \quad \square$$

Now we can prove that $K_{3,3}$ is non-planar.

Proposition 12. *$K_{3,3}$ is* not *a planar graph.*

Proof. $K_{3,3}$ is a connected graph with $6 \geq 3$ many vertices which has 9 edges. Also, since $K_{3,3}$ is bipartite, it is triangle-free. Thus, if $K_{3,3}$ happened to be planar, then Theorem 9 would tell us that $9 \leq 2 \cdot 6 - 4 = 8$; a contradiction. $\square$

It turns out that K_5 and $K_{3,3}$ are "morally" the only non-planar graphs. We will discuss exactly what "morally" means next time.

A bit of formalism. Let $G = (V, E)$ be a graph. A *planar embedding* of G is a collection of functions $f\colon V \to \mathbb{R}^2$ and $g_e\colon [0,1] \to \mathbb{R}^2$ for each $e \in E$ with the following properties:

- f is an injection.
- g_e is an injection and is continuous for each $e \in E$.
- For each $e \in E$, if $e = uv$, then $\{g_e(0), g_e(1)\} = \{f(u), f(v)\}$
- For every $e \neq s \in E$, we have $x \in g_e([0,1]) \cap g_s([0,1])$ if and only if $x = f(v)$ where $e \cap s = \{v\}$.

In other words, f tells you where to place the vertices, and g_e tells you how to draw the edge e.

A planar graph is a graph for which there exists a planar embedding. A plane graph is a graph equipped with some fixed planar embedding.

The main fact which needs to be used to justify the definitions of faces and such is the Jordan curve theorem.

A *simple closed curve* in $\mathbb{R}^2$ is the set $\phi([0,1])$ where $\phi\colon [0,1] \to \mathbb{R}^2$ is a function satisfying:

- ϕ is continuous, and
- $\phi(x) = \phi(y)$ if and only if $\{x, y\} \subseteq \{0, 1\}$.

Informally, a simple closed curve is a closed loop in the plane without self-intersections.

Theorem 13 (Jordan Curve Theorem)**.** *Let C be a simple closed curve in $\mathbb{R}^2$. Then $\mathbb{R}^2 \setminus C = A \sqcup B$ where*

- *A and B are both connected (in the topological sense), and*
- *A is bounded and B is unbounded, and*
- *$\partial A = \partial B = C$.*

In other words, the curve C separates $\mathbb{R}^2$ into two pieces, and bounded "interior" region and an unbounded "exterior" region.