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Recall the following "trees are everywhere" theorem:

Theorem 1 (Trees are everywhere)**.** *Let T be any tree on t vertices. If G is any graph with $\delta(G) \geq t-1$, then G contains a copy of T.*

This theorem was proved by "greedily embedding" the tree vertex-by-vertex. Also, the theorem is tight for all $t \geq 2$, since K_{t-1} has only $t-1$ vertices and has $\delta(K_{t-1}) = t-2$.

Let's prove a couple related theorems about when we can embed a tree into a graph.

Theorem 2. *Let T be any tree on t vertices. If G is any graph with $\chi(G) \geq t$, then G contains a copy of T.*

Again, this theorem is tight for all $t \geq 2$ since K_{t-1} has only $t-1$ vertices and has $\chi(K_{t-1}) = t-1$.

Before we dive into the proof, let's introduce some terminology:

Definition 3. *Let t be a positive integer. A graph G is said to be t-*critical *if $\chi(G) \geq t$ yet every proper subgraph H of G has $\chi(H) \leq t-1$. That is, G is minimal with respect to the property of having $\chi(G) \geq t$.*

Note that the only 1-critical graph (up to isomorphism) is K_1 (whether or not you consider the null-graph to be a thing). Also, it is pretty quick to verify that the only 2-critical graph (up to isomorphism) is K_2 (this is a good, quick exercise to help you understand the definition). Finally, G is 3-critical if and only if G is an odd-cycle; this is a homework exercise. There is no good classification of t-critical graphs for any $t \geq 4$.

t-critical graphs are very useful for proving various statements of the form "If G has property BLAH_1, then $\chi(G) \leq \text{BLAH}_2$". Indeed, it is often the case that property BLAH_1 is maintained under taking subgraphs, and:

Proposition 4. *If G has $\chi(G) \geq t$, then G contains a subgraph which is t-critical.*

Proof. Let $\mathcal{G}$ denote the set of all subgraphs H of G with $\chi(H) \geq t$. Note that $\mathcal{G}$ is non-empty since $G \in \mathcal{G}$. Thus, among all elements of $\mathcal{G}$, let H be one which minimizes $|V(H)| + |E(H)|$. We claim that H is t-critical. Indeed, let H' be any proper subgraph of H. Since H' is a proper subgraph of H, we must have $|V(H')| + |E(H')| < |V(H)| + |E(H)|$. Thus, by the definition of H, we must have $H' \notin \mathcal{G}$ and so $\chi(H) \leq t-1$. □

Another useful observation about t-critical graphs is that they are "dense":

Proposition 5. *If G is t-critical, then $\chi(G) = t$ and $\delta(G) \geq t-1$.*

Proof. We have already mentioned the case of 1-critical graphs, so suppose that $t \geq 2$. Since $\chi(G) \geq t \geq 2$, we know that G has at least two vertices. Let $v \in V(G)$ be such that $\deg v = \delta(G)$ and set $H = G - v$, which is a proper subgraph of G. Since G is t-critical, we know that $\chi(H) \leq t-1$. Thus, let $f\colon V(H) \to [t-1]$ be a proper coloring of H.

1. We show first that $\chi(G) = t$. Indeed, extend f to a coloring of G by defining $f(v) = t$. Since $V(G) = V(H) \sqcup \{v\}$, certainly f is a proper t-coloring of G since color t is un-used in $V(H)$. Thus $\chi(G) \leq t \implies \chi(G) = t$ since we already have $\chi(G) \geq t$ by assumption.

2. Suppose for the sake of contradiction that $\delta(G) \le t-2$, so $\deg v \le t-2$. But then there is some color $c \in [t-1]$ which is un-used in $N(v) \subseteq V(H)$ since $|N(v)| \le t-2$. Defining $f(v) = c$, we arrive at a proper $(t-1)$-coloring of G; a contradiction since $\chi(G) \ge t$. □

The following observation is not really relevant to the proof of Theorem 2, but it is good to know:

Proposition 6. *If G is t-critical, then G is connected.*

Proof. Suppose that G is disconnected and has connected components $G_1, \dots, G_k$ for some $k \ge 2$. Observe that $\chi(G) = \max_{i\in[k]} \chi(G_i)$ (why?) and so there is some $i \in [k]$ for which $\chi(G_i) = \chi(G) \ge t$ (really, equals t thanks to Proposition 5, but this is unimportant here). But then G_i is a proper subgraph of G with $\chi(G_i) \ge t$, contradicting the fact that G is t-critical. □

With the definition of t-critical graphs and the properties we just proved, the proof of Theorem 2 is basically one line (if that one line is long enough)!

Proof of Theorem 2. Since $\chi(G) \ge t$, we can find a t-critical subgraph H in G thanks to Proposition 4. Then, thanks to Proposition 5, this H has $\delta(H) \ge t-1$. Thus, H contains a copy of T by Theorem 1 and so G also has a copy of T. □

Let's now prove the following "Ramsey-type" result. We will discuss Ramsey's theorem toward the end of the class. For now, a "Ramsey-type" result is any statement of the form: "Either G has property BLAH_1 or $\overline{G}$ has property BLAH_2 (or both)". We've encountered Ramsey-type results before, e.g.:

- Either G or $\overline{G}$ is connected.
- If G has n vertices, then either G or $\overline{G}$ has chromatic number at least $\sqrt{n}$.

Theorem 7 (Chvátal's Ramsey-type theorem)**.** *Let T be any tree on t vertices, let G be any graph on n vertices and let m be a positive integer. If $n \ge (t-1)(m-1)+1$, then either G contains a copy of T or $\overline{G}$ contains a copy of K_m (or both).*

Proof. If $m = 1$, then the claim is trivial since every graph contains a copy of K_1; thus we may suppose that $m \ge 2$.

Suppose that $\overline{G}$ does not contain a copy of K_m; we need to show that G contains a copy of T. Since $\overline{G}$ does not contain a copy of K_m, we know that $\omega(\overline{G}) \le m-1$. Of course $\omega(\overline{G}) = \alpha(G)$, so we bound

$$\chi(G) \ge \frac{n}{\alpha(G)} \ge \frac{(t-1)(m-1)+1}{m-1} = t-1+\frac{1}{m-1} > t-1.$$

Since $\chi(G)$ and $t-1$ are both integers, this implies that $\chi(G) \ge t$. Thus, G contains a copy of T thanks to Theorem 2 □

Chvátal's theorem is tight for all $t, m \ge 2$ (the statement if $t = 1$ or $m = 1$ is trivial). Indeed, if $n = (t-1)(m-1)$, consider forming G by taking $m-1$ disjoint copies of K_{t-1}. Since trees are connected and each connected component of G has only $t-1$ vertices, we see that G does not contain a copy of T. On the other hand, $\overline{G}$ is isomorphic to $K_{\underbrace{t-1,\dots,t-1}_{m-1}}$, which is $(m-1)$-partite and thus cannot contain a copy of K_m.

We turn now to discussing edge-colorings of a graph. Much like for vertex-colorings, an edge-coloring is nothing more than a function $f\colon E(G) \to C$ where C is some set of colors. An edge coloring is said to be proper if $f(e) \neq f(s)$ whenever e and s share a common vertex. In other words, proper edge-colorings of G are exactly proper vertex-colorings of the line graph $L(G)$.

Definition 8. *The* edge-chromatic number *or* chromatic index *of G, denoted by $\chi'(G)$, is the smallest integer t for which G has a proper t-edge-coloring. Equivalently, $\chi'(G) = \chi(L(G))$.*

Note that $\chi'(G) = 0$ if and only if G has no edges.

Let $f\colon E(G) \to C$ be a proper edge-coloring of G. Then the color classes of f are matchings in G. In other words, proper edge-colorings are equivalent to partitions of the edges into matchings (if you don't care about the actual nature of the colors). To re-iterate this:

Observation 9. *$\chi'(G)$ is the smallest integer t for which we can partition $E(G)$ into t matchings.*

Let's derive a couple lower bounds on $\chi'(G)$.

Proposition 10. *For any graph G,*

$$\chi'(G) \geq \Delta(G), \quad \textit{and} \quad \chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}.$$

Proof. For each $v \in V(G)$, each of the $\deg v$-many edges incident to v must receive different colors; thus $\chi'(G) \geq \Delta(G)$.

By definition, we can partition $E(G) = M_1 \sqcup \cdots \sqcup M_{\chi'(G)}$ where each M_i is a matching in G. Since $\alpha'(G)$ is the size of a largest matching, we know that $|M_i| \leq \alpha'(G)$ and so

$$|E(G)| = \sum_{i=1}^{\chi'(G)} |M_i| \leq \sum_{i=1}^{\chi'(G)} \alpha'(G) = \chi'(G)\alpha'(G) \implies \chi'(G) \geq \frac{|E(G)|}{\alpha'(G)}. \qquad \square$$

What about upper bounds?

Proposition 11. *If G is any graph with at least one edge, then $\chi'(G) \leq 2\Delta(G) - 1$.*

Proof. We have $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1$ by our greedy-coloring argument applied to the line graph.

Now, for any edge $uv \in E(G)$, observe that $\deg_{L(G)} uv = \deg_G u + \deg_G v - 2$, and so

$$\chi'(G) \leq \Delta(L(G)) + 1 = 1 + \max_{uv \in E(G)} \big(\deg_G u + \deg_G v - 2\big) \leq 1 + \big(2\Delta(G) - 2\big) = 2\Delta(G) - 1. \qquad \square$$

Therefore, provided G has some edges,

$$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1.$$

It turns out that the possible range for χ' is even smaller than this!

Theorem 12 (Vizing's Theorem)**.** *If G is any graph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

So there are only two options for the edge-chromatic number! Although this is the case, it is still (generally) a difficult task to determine whether $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.

A proof of Vizing's theorem would be a bit too involved for us at the moment, so we won't prove it here. Also, one quick remark: everything we've said so far about χ' works for multigraphs as well *except* for Vizing's theorem, which requires simple graphs. Vizing's theorem for multigraphs states that $\Delta(G) \le \chi'(G) \le \Delta(G) + \mu(G)$ where $\mu(G)$ is the largest multiplicity of an edge of G (so $\mu(G) = 1$ iff G is simple).

Let's compute a couple edge-chromatic numbers.

Theorem 13. *For each integer $n \ge 2$,*

$$\chi'(K_n) = \begin{cases} n-1 & n \text{ is even,} \\ n & n \text{ is odd.} \end{cases}$$

Proof. (Lower bounds): We already know that $\chi'(K_n) \ge \Delta(K_n) = n - 1$ always.

Now, if n is odd, then $\alpha'(K_n) = (n-1)/2$, and so

$$\chi'(K_n) \ge \frac{|E(K_n)|}{\alpha'(K_n)} = \frac{\binom{n}{2}}{\frac{n-1}{2}} = \frac{\frac{n(n-1)}{2}}{\frac{n-1}{2}} = n.$$

(Upper bounds): It suffices to prove only that $\chi'(K_n) \le n - 1$ whenever n is even. Indeed, if n is odd, then we would have

$$\chi'(K_n) \le \chi'(K_{n+1}) \le (n+1) - 1 = n,$$

since K_n is a subgraph of K_{n+1} and $n + 1$ is even.

If $n = 2$, then certainly $\chi'(K_2) = 2 - 1 = 1$ since K_2 has exactly one edge; thus we may suppose that $n \ge 4$ is even.

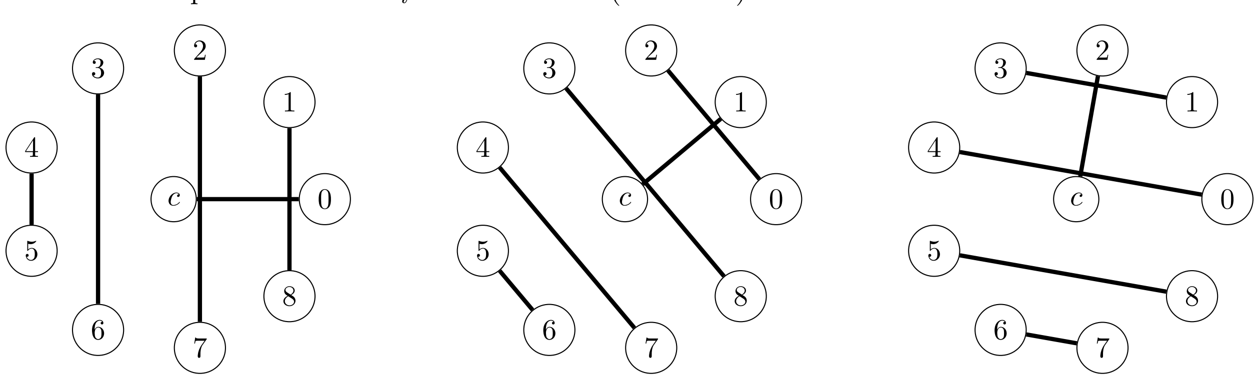
Set $m = n - 1$, so $m \ge 3$ is odd. We consider labeling the vertex-set of K_n as $V(K_n) = \{c\} \sqcup \{0, \ldots, m-1\}$; imagine c as a center vertex and the rest of the m vertices arranged on a circle around c. When defining the coloring, all arithmetic will be done modulo m; e.g. $-x = m - x$.

For each $i \in \{0, \ldots, m-1\}$, we define

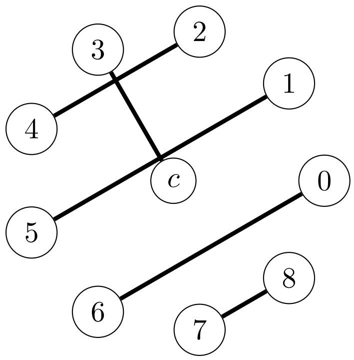
$$M_i = \Big\{\{c, i\}\Big\} \cup \Big\{\{i + x, i - x\} : x \in [m-1]\Big\}$$

Note that some edges are listed multiple times when defining M_i for convenience; we, of course, take each of these edges only once.

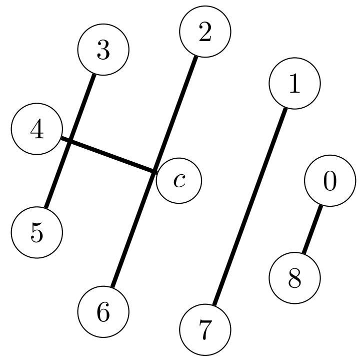
Below is a picture of the M_i's when $n = 10$ (so $m = 9$).



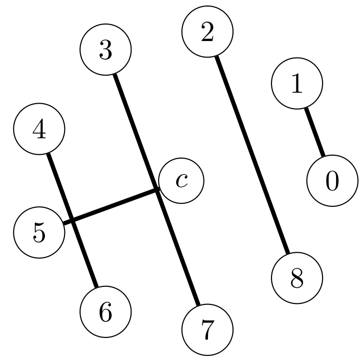
M_0 M_1 M_2



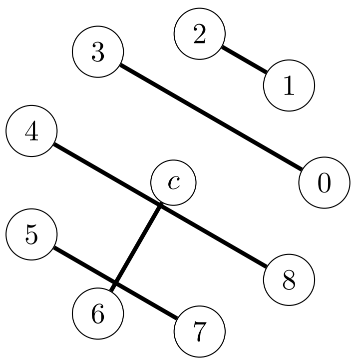
M_3



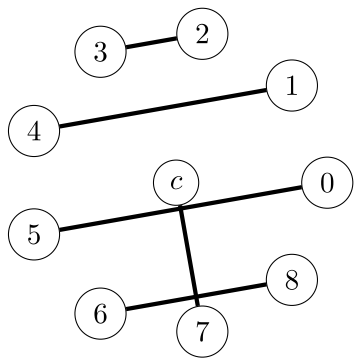
M_4



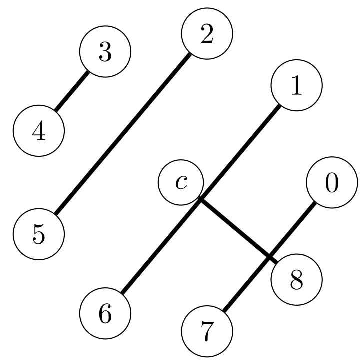
M_5



M_6



M_7



M_8

If we can show that $M_0, \ldots, M_{m-1}$ are matchings which cover every edge of K_n, then we will have shown that $\chi'(G) \leq m = n - 1$ as needed.

Let's show first that M_i is a matching for each $i \in \{0, \ldots, m-1\}$. Fix any $x, y \in [m-1]$, we must show that $\{i+x, i-x\}$ and $\{i+y, i-y\}$ are either disjoint or equal (i.e. they don't intersect in a single vertex). Certainly if $i+x = i+y$ or $i-x = i-y$, then $x = y$ and so these are the same edge. Thus, suppose that $i+x = i-y$ or $i-x = i+y$; both of these cases imply that $x+y = 0$ (arithmetic modulo m). In other words $x = -y$, which implies $\{i+x, i-x\} = \{i-x, i+x\} = \{i+y, i-y\}$ as needed.

Now we need to show that every edge of K_n is contains in one of these matchings. Fix any edge $xy \in E(K_n)$. If (wlog) $x = c$, then $y \in \{0, \ldots, m-1\}$ and so $xy \in M_y$. Thus, we just need to consider the case where $x, y \in \{0, \ldots, m-1\}$. Since m is odd (and thus 2 and m are coprime), we can find some $i \in \{0, \ldots, m-1\}$ such that $2i = x + y$ (again, arithmetic modulo m). Now, $x \neq y \in \{0, \ldots, m-1\}$ and so either $x - i \neq 0$ or $y - i \neq 0$; without loss of generality $x - i \neq 0$. Set $z = x - i$. Since $z \in [m-1]$, we know that $\{i+z, i-z\} \in M_i$. Of course, $i + z = i + (x - i) = x$ and $i - z = i - (x - i) = 2i - x = y$, so $xy \in M_i$, which concludes the proof. $\square$