These notes are from https://mathematicaster.org/teaching/graphs2022/extra_03-31.pdf

We begin by proving what is arguably the most important theorem about matchings: Hall's marriage theorem. This theorem specifically concerns matchings in bipartite graphs. First, note that if G is a bipartite graph with parts A, B, then $\alpha'(G) \le \min\{|A|, |B|\}$. Additionally, G has a matching which saturates, say, A if and only if $\alpha'(G) = |A|$.

For a set $S \subseteq V(G)$, we use $N(S)$ to denote the union of the neighborhoods of all elements of S — that is $N(S) = \bigcup_{s \in S} N(s)$. Note that $t \in N(S)$ if and only if there is some $s \in S$ such that $st \in E(G)$.

Theorem 1 (Hall's Marriage Theorem)**.** *Let G be a bipartite graph with parts A and B. Then G has a matching which saturates A (i.e. $\alpha'(G) = |A|$) if and only if $|N(S)| \ge |S|$ for all $S \subseteq A$.*

Note that the condition $|N(S)| \ge |S|$ is trivial if $S = \varnothing$, so one needs only consider non-empty subsets of A in practice.

Hall's marriage theorem is another "the obvious necessary condition is sufficient". Indeed, intuitively, in order to be able to match every element of A, each subset of A needs to have enough "potential matches" available.

Proof. ($\Rightarrow$) Suppose that $M \subseteq E(G)$ is a matching which saturates A; we build a function $f\colon A \to B$ where $f(a) = b$ if $ab \in M$. The function f is well-defined since M saturates A and G is bipartite (so M matches every vertex in A to some vertex in B). Also f is an injection since M is a matching and so no two vertices of A are matched to the same vertex in B. Furthermore, $f(a) \in N(a)$ for every $a \in A$. Thus, for any subset $S \subseteq A$, we have $f(S) \subseteq N(S)$ and so $|S| = |f(S)| \le |N(S)|$.

($\Leftarrow$) This is the interesting direction. Since G is bipartite with parts A, B, note that G has a matching which saturates A if and only if $\alpha'(G) = |A|$.

We prove the contrapositive, so we show that if $\alpha'(G) < |A|$, then there is some $S \subseteq A$ for which $|S| > |N(S)|$. By Kőnig's theorem, we know that $\alpha'(G) = \beta(G)$ and so also $\beta(G) < |A|$. As such, we can find a vertex-cover $C \subseteq V(G)$ with $|C| = \beta(G) < |A|$. Observe that

$$|A| > |C| = |C \cap A| + |C \cap B| \quad \implies \quad |C \cap B| < |A| - |C \cap A| = |A \setminus C|. \tag{1}$$

Now, set $S = A \setminus C$, so $S \subseteq A$. Consider any $b \in N(S)$, so there is some $s \in S$ with $sb \in E(G)$. Since C is a vertex-cover of G, C contains one of s or b. However, $s \notin C$ since $s \in S = A \setminus C$, so $b \in C$. Since this holds for all $b \in N(S)$, we have found that $N(S) \subseteq C$. Of course, $N(S) \subseteq B$ and so $N(S) \subseteq C \cap B$. Applying (1) then yields,

$$|N(S)| \le |C \cap B| < |A \setminus C| = |S|,$$

and so S is the subset we're looking for. ☐

We begin with a nice application.

Theorem 2. *If G is a k-regular, bipartite graph for some $k \ge 1$, then G has a perfect matching.*

Proof. Call the two parts A, B. We show first that $|A| = |B|$. Indeed, we apply the bipartite handshaking lemma to find that

$$k|A| = \sum_{a \in A} \deg a = \sum_{b \in B} \deg b = k|B| \implies |A| = |B|,$$

since $k \neq 0$. Thus, we just need to show that G has a matching which saturates A; this will imply that $\alpha'(G) = |A| = |B|$ and so this matching is actually a perfect matching.

Fix any non-empty $S \subseteq A$; we must show that $|N(S)| \geq |S|$. Consider the subgraph G' of G induced on $S \cup N(S)$. Certainly $\deg_{G'} a = \deg_G a = k$ for all $a \in S$, and $\deg_{G'} b \leq \deg_G b = k$ for all $b \in N(S)$. We apply the bipartite handshaking lemma to G' and the fact that $k > 0$ to find that

$$k|S| = \sum_{a \in S} \deg_{G'} a = \sum_{b \in N(S)} \deg_{G'} b \leq k|N(S)| \implies |S| \leq |N(S)|.$$

□

In fact, we can extend the above theorem.

Theorem 3. *Let G be a graph with parts A, B and suppose that no vertex of A is isolated. If $\deg a \geq \deg b$ whenever $a \in A$, $b \in B$ and $ab \in E(G)$, then G has a matching which saturates A.*

Before diving into the proof, we mention two useful facts:

- Silly sizes: If X is a non-empty finite set, then $|X| = \sum_{x \in X} 1$ and $1 = \sum_{x \in X} \frac{1}{|X|}$.
- Switching the order of summation: Suppose that X, Y are finite sets and $\Omega \subseteq X \times Y$. For any function $f \colon X \times Y \to \mathbb{R}$,

$$\sum_{(x,y) \in \Omega} f(x,y) = \sum_{x \in X} \sum_{\substack{y \in Y: \\ (x,y) \in \Omega}} f(x,y) = \sum_{y \in Y} \sum_{\substack{x \in X: \\ (x,y) \in \Omega}} f(x,y).$$ [1]

Our use of the second fact will be as follows: for any $S \subseteq A$,

$$\sum_{a \in S} \sum_{b \in N(a)} f(a,b) = \sum_{b \in N(S)} \sum_{a \in N(b)} f(a,b).$$

This is seen by taking $X = S$, $Y = N(S)$ and $\Omega = \{(a,b) \in S \times N(S) : ab \in E(G)\}$.

Proof. First note that $|N(a)| = \deg a \geq 1$ for all $a \in A$ since no vertex of A is isolated.

We verify Hall's condition, so fix any $S \subseteq A$; we must show that $|N(S)| \geq |S|$.

$$\begin{aligned} |S| = \sum_{a \in S} 1 = \sum_{a \in S} \sum_{b \in N(a)} \frac{1}{\deg a} &= \sum_{b \in N(S)} \sum_{a \in N(b)} \frac{1}{\deg a} \\ &\leq \sum_{b \in N(S)} \sum_{a \in N(b)} \frac{1}{\deg b} = \sum_{b \in N(S)} 1 = |N(S)|, \end{aligned}$$

where the inequality follows from the assumption since $b \in N(S) \subseteq B$, $a \in N(b) \subseteq A$ and $ab \in E(G)$. □

Here's a nice corollary that's useful to keep in mind:

Corollary 4. *Let G be a bipartite graph with parts A, B and fix an integer $k \geq 1$. If $\deg a \geq k$ for all $a \in A$ and $\deg b \leq k$ for all $b \in B$, then G has a matching which saturates A.*

[1] One possible proof of this fact is accomplished considering $\sum_{x \in X, y \in Y} g(x,y)$ where $g(x,y) = f(x,y)$ if $(x,y) \in \Omega$ and $g(x,y) = 0$ otherwise. This fact can be extended to the case when X and Y are (countably) infinite, but one needs some extra assumptions on the function f in order to do so.

Hall's theorem is often applied to objects other than graphs. One common situation is when one wishes to select objects from a collection of sets without repetition.

Definition 5. *For finite sets $S_1, \ldots, S_n$, a* system of distinct representatives *(SDR) is a collection of* distinct *elements $s_1, \ldots, s_n$ such that $s_i \in S_i$ for all $i \in [n]$. The elements $s_1, \ldots, s_n$ are referred to as* representatives.

If we didn't require that the representatives were distinct, then we would only need to require that each set was non-empty. However, the distinctness throws in some complications.

Theorem 6 (Hall's theorem for SDRs)**.** *For an integer $n \geq 1$, let $S_1, \ldots, S_n$ be finite (possibly empty) sets. There exists a system of distinct representatives for these sets if and only if*

$$|I| \leq \left|\bigcup_{i \in I} S_i\right|,$$

for every $I \subseteq [n]$.

Proof. We build a bipartite graph G with parts $A = [n]$ and $B = \bigcup_{i=1}^n S_i$ where $ab \in E(G)$ $(a \in A, b \in B)$ if and only if $b \in S_a$. Then there exists a system of distinct representatives if and only if G has a matching which saturates A. Now, for any $I \subseteq A = [n]$, we observe that

$$N(I) = \bigcup_{i \in I} S_i,$$

and so the condition in the theorem is equivalent to Hall's theorem applied to the graph G. □

One useful observation is the following rephrasing of Corollary 4 to SDRs.

Corollary 7. *Let $S_1, \ldots, S_n$ be finite sets and fix an integer $k \geq 1$. If $|S_i| \geq k$ for all $i \in [n]$ and each element of $\bigcup_{i=1}^n S_i$ is contained in at most k of the S_i's, then there exists a system of distinct representatives.*

One fun application of this corollary is that one can always extend a Latin rectangle to a Latin square. I'll make this a worksheet question in our next discussion session :)

Hall's marriage theorem is excellent since it tells us exactly when there is a matching which saturates one side of G. But what if we just want to know the size of the largest matching?

Theorem 8 (Hall's Marriage Theorem, extended)**.** *Let G be a bipartite graph with parts A, B. For a subset $S \subseteq A$, define* $\text{defect}(S) = \max\{0, |S| - |N(S)|\}$*. Then*

$$\alpha'(G) = |A| - \max_{S \subseteq A} \text{defect}(S).$$

Notice that Hall's condition is that $\text{defect}(S) = 0$ for all $S \subseteq A$.

Proof. We prove first that $\alpha'(G) \geq |A| - \max_{S \subseteq A} \text{defect}(S)$. To this end, set $d = \max_{S \subseteq A} \text{defect}(S)$. We build a new graph G' by adding d new vertices to B and connecting each of them to all of A; call these new vertices B'. Then, for any $S \subseteq A$, we have $N_{G'}(S) = N_G(S) \sqcup B'$ and so

$$|N_{G'}(S)| = |N_G(S)| + d \geq |N_G(S)| + \max\{0, |S| - |N_G(S)|\} \geq |S|.$$

Thus, we may apply Hall's marriage theorem to G' to find a matching which saturates A. By then deleting the vertices in B', we are left with a matching of G which has at least $|A| - d$ many edges.

We now prove that $\alpha'(G) \leq |A| - \max_{S \subseteq A} \operatorname{defect}(S)$. Let $M \subseteq E(G)$ be a maximum matching of G, let $A_{in} \subseteq A$ be the set of vertices of A covered by M and let $A_{out} \subseteq A$ be the set of vertices in A not covered by M. Of course, $A = A_{in} \sqcup A_{out}$. Much like in our proof of Hall, we build a function $f\colon A_{in} \to B$ where $f(a) = b$ iff $ab \in M$. Just like earlier, f is an injection and $f(a) \in N(a)$ for all $a \in A_{in}$.

Now, consider any $S \subseteq A$; we have $f(S \cap A_{in}) \subseteq N(S \cap A_{in})$ and so $|S \cap A_{in}| = |f(S \cap A_{in})| \leq |N(S \cap A_{in})| \leq |N(S)|$. Therefore, $|S \cap A_{out}| = |S| - |S \cap A_{in}| \geq |S| - |N(S)|$. Since also $|S \cap A_{out}| \geq 0$, we have shown that $|S \cap A_{out}| \geq \operatorname{defect}(S)$. In particular, $|A_{out}| \geq \max_{S \subseteq A} |S \cap A_{out}| \geq \max_{S \subseteq A} \operatorname{defect}(S)$. We conclude that

$$\alpha'(G) = |M| = |A_{in}| = |A| - |A_{out}| \leq |A| - \max_{S \subseteq A} \operatorname{defect}(S). \qquad \square$$

To end things off, we used Kőnig to prove Hall (and thus extended Hall); let's show that Hall (specifically the extended version) additionally implies Kőnig. In other words, Kőnig and Hall are morally the same theorem.

Hall implies Kőnig. We proved last time that $\beta(G) \geq \alpha'(G)$ always (G doesn't even need to be bipartite here), so we need only prove the reverse inequality; i.e. $\beta(G) \leq \alpha'(G)$.

Fix any $S \subseteq A$ and set $C = (A \setminus S) \sqcup N(S)$; we claim that C is a vertex-cover of G. Indeed, fix any edge $ab \in E(G)$ with $a \in A$ and $b \in B$. If $a \notin S$, then $a \in A \setminus S \subseteq C$ and so C covers ab. If $a \in S$, then $b \in N(a) \subseteq N(S) \subseteq C$ and so C covers ab. Therefore,

$$\beta(G) \leq |C| = |A \setminus S| + |N(S)| = |A| - |S| + |N(S)|,$$

for every $S \subseteq A$. Additionally, A is clearly a vertex-cover of G and so $\beta(G) \leq |A|$. Putting these bounds together, we have

$$\begin{aligned} \beta(G) &\leq \min_{S \subseteq A} \min\bigl\{|A|, |A| - |S| + |N(S)|\bigr\} = \min_{S \subseteq A}\bigl(|A| - \max\{0, |S| - |N(S)|\}\bigr) \\ &= \min_{S \subseteq A}\bigl(|A| - \operatorname{defect}(S)\bigr) = |A| - \max_{S \subseteq A} \operatorname{defect}(S). \end{aligned}$$

Finally, the extended version of Hall's theorem (Theorem 8) states that $\alpha'(G) = |A| - \max_{S \subseteq A} \operatorname{defect}(S)$ and so $\beta(G) \leq \alpha'(G)$ as needed. $\square$