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Here is a proof of Menger's theorem which is slightly different from that in the book. The main workhorse of the proof is Lemma 1 and it will be our friend later in the course as well :)

Let G be a graph and fix any $A, B \subseteq V(G)$ with A, B non-empty (note that A and B may intersect). An A-B path is a path $(v_0, \ldots, v_k)$ in G such that $v_0 \in A$, $v_k \in B$ and none of the internal vertices $v_1, \ldots, v_{k-1}$ live in either A or B. Note that if $a \in A \cap B$, then (a) is an A-B path.

An A-B separator is a set $S \subseteq V(G)$ such that $G - S$ contains no $(A \setminus S)$-$(B \setminus S)$ path.[1] Note that $S = A$ and $S = B$ are trivially A-B separators; in particular, A-B separators exist. Let $\kappa_G(A, B)$ denote the size of a minimum A-B separator in G.

Observe that S is an A-B separator if and only if every A-B path in G contains a vertex from S. In particular, if $P_1, \ldots, P_k$ are vertex-disjoint A-B paths, then $k \leq \kappa_G(A, B)$. Let $p_G(A, B)$ denote the maximum number of vertex-disjoint A-B paths in G.

Lemma 1. *Let G be a graph and fix any $A, B \subseteq V(G)$ with A, B non-empty. Then $p_G(A, B) = \kappa_G(A, B)$.*

Often, a collection of vertex-disjoint A-B paths in G is called an A-B connector. With this language, the lemma states that any maximum A-B connector has the same size as any minimum A-B separator.

Proof. We have already noted that $p_G(A, B) \leq \kappa_G(A, B)$, so we must prove that $p_G(A, B) \geq \kappa_G(A, B)$. We prove this by induction on $|E(G)|$.

For the base case, consider when $|E(G)| = 0$. Then every A-B path has the form (x) for some $x \in A \cap B$ (should such an x exist). Thus, $p_G(A, B) = |A \cap B| = \kappa_G(A, B)$.

Suppose now that $|E(G)| \geq 1$ and fix any $e \in E(G)$. Set $k = \kappa_G(A, B)$; we must find k vertex-disjoint A-B paths in G in order to show that $p_G(A, B) \geq k$. Consider the graph $G - e$. It must be the case that $\kappa_{G-e}(A, B) \in \{k, k-1\}$ (why? It's the same reasoning as in HW7.3.2). If $\kappa_{G-e}(A, B) = k$, then $p_{G-e}(A, B) = k$ by the induction hypothesis and so there are k vertex-disjoint A-B paths in $G - e$. Each of these paths is also a path in G and so $p_G(A, B) \geq k$ as needed.

Thus, suppose that $\kappa_{G-e}(A, B) = k - 1$. Let S be a minimum A-B separator in $G - e$, so $|S| = k - 1$. By assumption, $\kappa_G(A, B) = k > |S|$ and so there is an A-B path in $G - S$. Since these paths don't exist in $(G - e) - S$, it must be the case that each of them use the edge e. Let P be any such path and label $e = v_1 v_2$ such that P uses the vertex v_1 before the vertex v_2. Observe that every A-$\{v_2\}$ path in $G - S$ and every $\{v_1\}$-B path in $G - S$ uses the edge e or else we could avoid e all-together.

Now, we know that $v_1, v_2 \notin S$. For $i \in \{1, 2\}$, set $S_i = S \cup \{v_i\}$; then $|S_i| = k$. Additionally, both S_1 and S_2 are A-B separators in G since every A-B path in $G - S$ used the edge $e = v_1 v_2$.

Let T_1 be a minimum A-S_1 separator in $G - e$ and let T_2 be a minimum S_2-B separator in $G - e$. We claim that T_1, T_2 are both A-B separators in G. Indeed, suppose for the sake of contradiction that there were an A-B path in $G - T_1$; call it P. Since S_1 is an A-B separator, the path P must contain a vertex from S_1, let P' be the sub-path of P which ends at the first vertex encountered in S_1. Then P' is an A-S_1 path in $G - T_1$, so since T_1 is an A-S_1 separator in $G - e$, this means that P' uses the edge e. Since $v_1 \in S_1$, the only way that this is possible is if P' ends in (v_2, v_1). But then P' sans the last vertex v_1 is an A-$\{v_2\}$ path in $G - S_1 = G - (S \cup \{v_1\})$ contradicting the fact

[1] The book also defines a notion of a separator which is a little bit different than this, so be careful.

that any A-$\{v_2\}$ path in $G - S$ must use the edge e. A symmetric argument works to show that T_2 is also an A-B separator.

Since T_1, T_2 are A-B separators in G, we must have $|T_i| \geq \kappa_G(A, B) = k$. Therefore, $\kappa_{G-e}(A, S_1) = |T_1| \geq k$ and $\kappa_{G-e}(S_2, B) = |T_2| \geq k$. By the induction hypothesis, we therefore know that

$$p_{G-e}(A, S_1) = \kappa_{G-e}(A, S_1) \geq k \qquad \text{and} \qquad p_{G-e}(S_2, B) = \kappa_{G-e}(S_2, B) \geq k.$$

Thus, there are at least k vertex-disjoint A-S_1 paths in $G - e$ and at least k vertex-disjoint S_2-B paths in $G - e$. Since $|S_1| = |S_2| = k$, there must actually be exactly k of each. Furthermore, by labeling $S = \{s_1, \ldots, s_{k-1}\}$, this same fact means that we can find vertex-disjoint A-S_1 paths $P_1, \ldots, P_k$ such that the last vertex of P_k is v_1 and, for each $i \in [k-1]$, the last vertex of P_i is s_i. Similarly, we can find vertex-disjoint S_2-B paths $P'_1, \ldots, P'_k$ such that the first vertex of P'_k is v_2 and, for each $i \in [k-1]$, the first vertex of P'_i is s_i. For each $i \in [k-1]$ note that concatenating P_i and P'_i is an A-B path in G. Also, since $v_1v_2 \in E(G)$, concatenating P_k and P'_k is also an A-B path in G. After all this work, we see that these k concatenated paths are vertex-disjoint and so $p_G(A, B) \geq k$ as needed. □

After all that work, we can finally prove Menger's Theorem. Recall that paths are said to be internally-disjoint if they share no vertices beyond their end-points.

Theorem 2 (Menger's Theorem for vertex-connectivity)**.** *Let G be a graph on at least two vertices. G is k-connected (i.e. $\kappa(G) \geq k$) if and only if there are at least k internally-disjoint u-v paths for every $u \neq v \in V(G)$.*

Proof. ($\Leftarrow$) We prove the contrapositive: suppose that G is not k-connected, so $\kappa(G) \leq k - 1$. We need to show that there are some $u \neq v \in V(G)$ for which there are at most $k - 1$ many internally-disjoint u-v paths in G.

If G is a clique, then $G \cong K_n$ for some $n \leq k$ since $\kappa(K_n) = n - 1$. Consider any $u \neq v \in V(G)$. Since $|V(G) \setminus \{u, v\}| = n - 2$ there are at most $n - 2$ internally-disjoint u-v paths which use some vertex other than u and v. Along with the path (u, v), this yields at most $n - 1 \leq k - 1$ many internally-disjoint u-v paths.

Now suppose that G is not a clique; thus there is some $U \subseteq V(G)$ with $|U| = \kappa(G) \leq k - 1$ such that $G - U$ is disconnected. Consider any u, v in different connected components of $G - U$. Then every u-v path in G must use some vertex in U and hence there are at most $|U| \leq k - 1$ many internally-disjoint u-v paths in G.

($\Rightarrow$) We proceed by induction on $|E(G)|$. The claim is immediate if $|E(G)| = 0$ since then $\kappa(G) = 0$; thus, suppose that $|E(G)| \geq 1$. Fix any $u \neq v \in V(G)$; we must find at least $\kappa(G)$ many internally-disjoint u-v paths in G.

Case 1: $uv \in E(G)$. By HW7.3.2, we know that $\kappa(G - uv) \geq \kappa(G) - 1$. Thus, the induction hypothesis allows us to conclude that there are at least $\kappa(G - uv) \geq \kappa(G) - 1$ many internally-disjoint u-v paths in $G - uv$. These paths along with the path (u, v) is then a collection of at least $\kappa(G)$ many internally-disjoint u-v paths in G.

Case 2: $uv \notin E(G)$. Let $S \subseteq V(G - \{u, v\}) = V(G) \setminus \{u, v\}$ be a minimum $N(u)$-$N(v)$ separator in $G - \{u, v\}$, so $|S| = \kappa_{G-\{u,v\}}(N(u), N(v))$. We claim that u and v are in different connected components of $G - S$. Indeed, since $uv \notin E(G)$ every u-v path in G contains an $N(u)$-$N(v)$ path. All of these are destroyed upon deleting S and so there is no u-v path in $G - S$. In particular, $G - S$ is disconnected and so $|S| \geq \kappa(G)$. Thus, appealing to Lemma 1, we have

$$p_{G-\{u,v\}}(N(u), N(v)) = \kappa_{G-\{u,v\}}(N(u), N(v)) \geq \kappa(G).$$

In particular, there are at least $\kappa(G)$ many vertex-disjoint $N(u)$-$N(v)$ paths in $G - \{u, v\}$. Appending u to the start and v to the end of each of these paths then yields at least $\kappa(G)$ many internally-disjoint u-v paths in G. □

There is also an edge-connectivity version of Menger's Theorem.

Theorem 3 (Menger's Theorem for edge-connectivity)**.** *Let G be a graph on at least two vertices. G is k-edge-connected (i.e. $\lambda(G) \geq k$) if and only if there are at least k edge-disjoint u-v paths for every $u \neq v \in V(G)$.*

We will need an intermediate lemma about line graphs.

Lemma 4. *Let G be a graph and fix any $u \neq v \in V(G)$. If $(e_0, \ldots, e_k)$ is a path in the line graph $L(G)$ with $u \in e_0$ and $v \in e_k$, then there is a u-v path in G using only edges from $\{e_0, \ldots, e_k\}$.*

Proof. Consider the spanning subgraph H of G which has only the edges $e_0, \ldots, e_k$. We claim that H contains a u-v path which will verify the claim.

Consider any partition $V(H) = V(G) = A \sqcup B$ with $u \in A$ and $v \in B$; we must show that H has an edge between A and B. Now, let $i \in \{0, \ldots, k\}$ be the smallest index such that $e_i \cap B \neq \varnothing$. We know that i exists since $v \in e_k \cap B$. We claim that $e_i \cap A \neq \varnothing$ as well which will conclude the proof. If $i = 0$, then $u \in e_i \cap A \implies e_i \cap A \neq \varnothing$, so suppose that $i \geq 1$. Then, since e_{i-1} and e_i are adjacent in $L(G)$, we must have $e_{i-1} \cap e_i \neq \varnothing$. By the definition of i, we know that $e_{i-1} \subseteq A$ and so the common vertex of e_{i-1} and e_i lives in A; thus $e_i \cap A \neq \varnothing$. □

Proof of Theorem 3. Note that $G \ncong K_1$ since it has at least two vertices; thus we don't need to worry about any edge-cases here.

($\Leftarrow$) We prove the contrapositive: suppose that G is not k-edge-connected, so $\lambda(G) \leq k - 1$. Then we can find an edge-cut $S \subseteq E(G)$ with $|S| = \lambda(G) \leq k-1$. Consider vertices u, v in different connected components of $G - S$, then every u-v path in G uses an edge from S and hence there are at most $|S| \leq k - 1$ many edge-disjoint u-v paths in G.

($\Rightarrow$) If $\lambda(G) = 0$, then the claim holds trivially, so we may suppose that $\lambda(G) \geq 1$. For $v \in V(G)$, define $E_v = \{e \in E(G) : e \ni v\}$ to be the set of edges incident to v. Since $|E_v| = \deg v \geq \delta(G) \geq \lambda(G) \geq 1$, we know that each E_v is non-empty.

Fix any $u \neq v \in V(G)$; we must find at least $\lambda(G)$ many edge-disjoint u-v paths in G. Consider the line graph $L(G)$ of G and consider the sets E_u and E_v (which are subsets of the vertices of $L(G)$). We claim that $\kappa_{L(G)}(E_u, E_v) \geq \lambda(G)$. Take any $S \subseteq E(G) = V(L(G))$ which is an E_u-E_v separator in $L(G)$. If $L(G) - S$ is connected, then either $S \supseteq E_u$ or $S \supseteq E_v$ which would mean that $|S| \geq \delta(G) \geq \lambda(G)$ as needed. Otherwise, $L(G) - S$ is disconnected. Now, $L(G - S) = L(G) - S$ and so DS1.7.4 implies that $G - S$ must be disconnected as well. Hence S is an edge-cut of G and so $|S| \geq \lambda(G)$ as needed.

We therefore know that $\kappa_{L(G)}(E_u, E_v) \geq \lambda(G)$ and so $p_{L(G)}(E_u, E_v) \geq \lambda(G)$ as well thanks to Lemma 1. Thus, there are at least $\lambda(G)$ vertex-disjoint E_u-E_v paths in $L(G)$. These paths correspond to disjoint sets of edges in G. While these sets of edges don't necessarily form u-v paths, each one contains a u-v path thanks to Lemma 4. Thus, we have found at least $\lambda(G)$ edge-disjoint u-v paths in G. □